Transform the Multivariate to Unitevariate System and Using Elimination and Duplication Matrices with Some Applications

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Abstract: *In this work, we have transformed the multivariate to unitevariate system, and studied the elimination and duplication matrices, using proposed approach depends on some algebra principle. And, a connection has been made between these two matrices with each of the (vec) and (v) operators for matrices.*

Several examples are given to illustrate the applications which is used to define and specialize the elimination and duplication matrices.

Keywords: ***Multivariate, Unitevariate, Elimination, Duplication.***

1. Introduction

The usefulness of the elimination and duplication matrices is demonstrated in mathematical statistics, matrix algebra and numerical analysis field [1].
If A is a square matrix of order n, its (1/2)n(n+1) supradiagonal elements are redundant. If we eliminate these elements from υecA (The column vector stacking the columns of A), this defines a new vector of order (1/2)n(n+1) which we denote as υ(A).
The matrix which, for arbitrary A, transforms υecA into υ(A) is the elimination matrix L, first mentioned by Tracy and Singh (1972) and later by Vetter (1975) and Balestra (1976) [2].
The inverse transformation from υ(A) to υecA. For lower triangular A, we shall see that L'υ(A) = υecA. We further introduce the duplication matrix D. The matrix D (or a matrix comparable to D) was previously defined by Tracy and Singh (1972), Browne (1974), Vetter (1975), Balestra (1976) and Nel (1978) [2].

The purpose of this paper, matrices L and D are used. Both matrices consist of zero and unite elements only, some necessary definitions and basic tools are given. Finally the multivariate system where the unknown matrix is symmetric matrix are solved and from any two squares matrices, symmetric matrix are find using L and D matrices.

2. Some Definitions

<u>Definitions(1)</u>: (Elimination matrix L) [2,3]

The ((1/2)n(n+1) , n^2) elimination matrix L performs for every (n , n) matrix A The transformation LυecA = υ(A).
The matrix form of L as: L = (L_{ij}) *i , j = 1, ..., n*, where:

$$L_{ij} = \begin{cases} 0_{(n-i+1)\times(i-1)} \quad \cdots \quad I_{(n-i+1)} & i = j \\ 0_{(n-i+1)\times(n)} & i \neq j \end{cases} \qquad i , j = 1, ..., n$$

An example, for n = 3 , is

$$L = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)_{6\times 9}$$

<u>Definitions(2)</u>: (Duplication matrix D) [2,3]

The $(n^2 , (1/2)n(n+1))$ duplication matrix D perform for every (n , n) matrix A The transformation $D\upsilon(A) = \upsilon ecA$.
The matrix form of D as: $D = (D_{ij})$ $i , j = 1, ..., n$, where:

$$D_{ij} = \begin{cases} \begin{pmatrix} 0_{(i-1)\times(n-i+1)} \\ I_{(n-i+1)} \end{pmatrix} & i = j \\ 0_{(n)\times(n-j+1)} & i < j \qquad\qquad i , j = 1, ..., n \\ \begin{pmatrix} 0_{(j-1)\times(n-j+1)} \\ E^{n-j+1}_{i-j+1} \end{pmatrix} & i > j \end{cases}$$

$E^{n}_{i\,j} = e^{n}_{i} \otimes e^{n'}_{j}$ and e^{n}_{i} is the ith unite vector of order (n × 1).
An example, for n = 3 , is

$$D = \left(\begin{array}{ccc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)_{9\times 6}$$

Definitions(3): (Matrix vector) [1,4]

Let $A = (a_{ij})$ be an $(m \times n)$ matrix and $A._j$ the j-th column of A; Then υecA is the $(mn \times 1)$ column vector. That is,

$$\upsilon ecA = \left(A._1 \;\; A._2 \;\; \cdots \;\; A._n\right)^T$$

Definitions(4): (Matrix half vector) [2]

If $A = (a_{ij})$ is a matrix of order $(n \times n)$ its $(1/2)n(n+1)$ supradiagonal elements are redundant. If we eliminate these elements from υecA, this defines a new vector of order $((1/2)n(n+1) \times 1)$ which we denote as $\upsilon(A)$ and called matrix half vector.

$$\upsilon(A) = \left(a_{11} \; a_{21} \; \cdots \; a_{n1} \, a_{22} \cdots a_{n2} \cdots a_{nn}\right)^T$$

3. Transform Multivariate System to Unitevariate System

Let A and B be known matrices of order $(n \times n)$ and X be an unknown matrix of order $(n \times n)$. The multivariate system is: [5,6]

$$AX = B \qquad \ldots(1)$$

where $A = (a_{ij})$, $B = (b_{ij})$ and $X = (x_{ij})$ $i , j = 1, ..., n,$

from equation (1) we obtain

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n\times n} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}_{n\times n} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}_{n\times n}$$

Then

$$\begin{pmatrix} a_{11}x_{11}+\cdots+a_{1n}\,x_{n1} & \cdots & a_{11}x_{1n}+\cdots+a_{1n}\,x_{nn} \\ a_{21}x_{11}+\cdots+a_{2n}\,x_{n1} & \cdots & a_{21}x_{1n}+\cdots+a_{2n}\,x_{nn} \\ \vdots & \ddots & \vdots \\ a_{n1}x_{11}+\cdots+a_{nn}\,x_{n1} & \cdots & a_{n1}x_{1n}+\cdots+a_{nn}\,x_{nn} \end{pmatrix}_{n\times n} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}_{n\times n} \quad \ldots(2)$$

From equation(2) we have the unitevariate system as follows

$$\begin{aligned} a_{11}x_{11}+\cdots+a_{1n}\,x_{n1} &= b_{11} \\ a_{21}x_{11}+\cdots+a_{2n}\,x_{n1} &= b_{21} \\ &\vdots \\ a_{n1}x_{11}+\cdots+a_{nn}\,x_{n1} &= b_{n1} \\ &\vdots \\ a_{11}x_{1n}+\cdots+a_{1n}\,x_{nn} &= b_{1n} \\ &\vdots \\ a_{n1}x_{1n}+\cdots+a_{nn}\,x_{nn} &= b_{nn} \end{aligned}$$

The above system can be written as matrix form:

$$Q\ \upsilon ecX = \upsilon ecB \quad \ldots(3)$$

where

$$Q = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A \end{pmatrix}_{n^2 \times n^2} ,\upsilon ecX = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}_{n^2 \times 1} ,\upsilon ecB= \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \\ \vdots \\ b_{1n} \\ \vdots \\ b_{nn} \end{pmatrix}_{n^2 \times 1}$$

And $A = (a_{ij})$ $i , j = 1, ..., n$.

That is the multivariate system (equation(1)) transform to unitevariate system (equation(3)).

Now applied the elimination and duplication matrices to find the solution of unitevariate system (equation(3)) using Vetter's solution (if the solution is exist)[2],

$$\upsilon ecX = D\ (L\ Q\ D)^{-1}\ L\ \upsilon ecB \qquad \ldots(4)$$

where (L Q D) is non singular.

In this paper, we find, the solution is exist if the unknown matrix X is symmetric. Also we find, for any two square matrices we can find symmetric matrix using the equation(4).

This is method will be illustrated by some examples in the next section.

4. Examples:

The performance of the proposed approach described in the above section will be tested it on the following examples.

Example (1):

Consider the following multivariate system: AX = B

Where:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix}_{3\times 3} \quad ,\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}_{3\times 3} \quad ,B = \begin{pmatrix} 3 & 1 & -8 \\ 4 & 1 & -4 \\ 3 & -5 & -9 \end{pmatrix}_{3\times 3}$$

and the unknown matrix X is symmetric.

Transform this system to unitevariate system as equation (3) we obtain:

$$Q\ \upsilon ecX = \upsilon ecB$$

where

$$Q=\begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 4 \end{pmatrix}_{9\times 9} \quad ,\upsilon ecX=\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}_{9\times 1} \quad \text{and}$$

$$\upsilon ecB=\begin{pmatrix} 3 \\ 4 \\ 3 \\ 1 \\ 1 \\ -5 \\ -8 \\ -4 \\ -9 \end{pmatrix}_{9\times 1}$$

At first, we find

LQD=

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&2&3&0&0&0&0&0&0\\ 0&2&1&0&0&0&0&0&0\\ -1&1&4&0&0&0&0&0&0\\ 0&0&0&1&2&3&0&0&0\\ 0&0&0&0&2&1&0&0&0\\ 0&0&0&-1&1&4&0&0&0\\ 0&0&0&0&0&0&1&2&3\\ 0&0&0&0&0&0&0&2&1\\ 0&0&0&0&0&0&-1&1&4 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$= \begin{pmatrix} 1&2&3&0&0&0\\ 0&2&1&0&0&0\\ -1&1&4&0&0&0\\ 0&0&0&2&1&0\\ 0&-1&1&4&0&0\\ 0&0&0&0&1&4 \end{pmatrix}_{6\times 6}$$

Then applied equation(4) and using Matlab, we have the unique solution :

$\upsilon ecX = \begin{pmatrix} -1 & 2 & 0 & 2 & 1 & -1 & 0 & -1 & -2 \end{pmatrix}^T$, then

$$X = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & -1 & -2 \end{pmatrix}_{3\times 3}$$

That is, the solution is symmetric matrix

<u>Example (2):</u>

Let us consider the following multivariate system of order 4:

$$\begin{pmatrix} -3 & 1 & 2 & 0 \\ -1 & 2 & -3 & 5 \\ 6 & 0 & 1 & -1 \\ 1 & -2 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} = \begin{pmatrix} 3 & 4 & -10 & -12 \\ 10 & -9 & 7 & 4 \\ 5 & 3 & 16 & 22 \\ 25 & -6 & 0 & 21 \end{pmatrix}$$

As example(1), Find

$$\text{L Q D} = \begin{pmatrix} -3 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -3 & 5 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 4 & 3 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 4 & 3 \end{pmatrix}_{10\times 10}$$

where Q matrix of order 16×16 as equation(3), L is elimination matrix of order 10×16 as definition(1) and D is duplication matrix of order 16×10 as definition(2).

Then using equation(4) and Matlab, to obtain the solution X:

$$X = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & -2 \\ 3 & 1 & -1 & 1 \\ 4 & -2 & 1 & 3 \end{pmatrix}$$

Example (3):

Find the symmetric matrix from the two matrices A and B of order 3×3 where:

$$A = \begin{pmatrix} 2 & 3 & -2 \\ 1 & 4 & -1 \\ -3 & 0 & 1 \end{pmatrix}_{3\times3} \quad \text{and} \quad B = \begin{pmatrix} 2 & 3 & -7 \\ 1 & -1 & -11 \\ -7 & -3 & -5 \end{pmatrix}_{3\times3}$$

To find the symmetric matrix X of order 3×3, we write the matrix form:

$$AX = B$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}_{3\times3}$$

Then, transform this system as equation(3), and applied equation(4) with using Matlab, we give symmetric matrix as:

$$X = \begin{pmatrix} 3 & 0 & 2 \\ 0 & -1 & -3 \\ 2 & -3 & 1 \end{pmatrix}_{3\times3}$$

Example (4):

Let the matrices A and B of order 4×4 as follows:

$$A = \begin{pmatrix} -3 & 1 & 2 & 0 \\ -1 & 2 & -3 & 5 \\ 6 & 0 & 1 & -1 \\ 1 & -2 & 4 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 5 \\ 3 & -2 & 6 & 1 \\ -4 & 2 & 3 & -3 \end{pmatrix}$$

At the same of example(3), we obtain the symmetric matrix X of order 4×4 :

$$X = \begin{pmatrix} 0.4355 & 2.0633 & 0.1217 & -0.2652 \\ 2.0633 & -21.9688 & -12.4485 & 1.9311 \\ 0.1217 & -12.4485 & -0.8869 & -6.1570 \\ -0.2652 & 1.9311 & -6.1570 & 8.5851 \end{pmatrix}_{4\times 4}$$

5. Conclusions

The main goal of this paper was to transform the multivariate system to unitevariate system and demonstrate that the elimination and duplication matrices is a powerful tool for solving the unitevariate system and finding symmetric matrix from any two square matrices. In additional to reduce the matrix of order $(n^2 \times n^2)$ to matrix of order $((1/2)n(n+1)\times(1/2)n(n+1))$.

Also, we have a diagonal matrix the element of diagonal is a matrix coefficient and we use the Matlab program (ver.6.5) to calculate the inverse matrix as well as some computation.

Finally, efficiency of this method for solving these problems has been approved by some examples.

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تحويل النظام المتعدد المتغيرات إلى أحادي المتغيرات واستخدام مصفوفتي الحذف والنسخ مع بعض التطبيقات

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المستخلص

في هذا العمل، حولنا النظام المتعدد المتغيرات إلى أحادي المتغيرات، ودرسنا مصفوفتي الحذف والنسخ، باستعمال أسلوب مقترح يعتمد على بعض مبادئ الجبر. والاتصال بين هاتين المصفوفتين بكل من عاملي (υec) و(υ) للمصفوفات.
عدة أمثلة أعطيت بشكل خاص لتوضيح التطبيقات التي استعملت لتعريف مصفوفتي الحذف والنسخ.