On S*-Convergence Nets And Filters

Sabiha I. Mahmood

Al-Mustansiriyah University

College of Science - Department of Mathematics

Abstract: *This paper is devoted to introduce and study many topological properties of s*-convergence of nets and s*-convergence of filters by using the concept of s*-open sets, also some properties of s*-cluster points of nets and filters has been studied.*

Key words: ***s*-open, s*-closed, s*-convergent, s*-cluster , s*-limit point and s*-irresolute***

1. Introduction

The concept of s*-closed set was first introduced by Al-Meklafi, S. [1] , by using the concept of semi-open set . Recall that a subset A of a topological space (X,τ) is called semi-open (briefly s-open) set if there exists an open subset U of X such that $U \subseteq A \subseteq cl(U)$.The complement of a semi-open set is defined to be semi-closed (briefly s-closed) [2] .Also, a subset A of a topological space (X,τ) is called s*-closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X [1] . The complement of an s*-closed set is defined to be s*-open .The family of all s*-open (resp.s*-closed) subsets of (X,τ) is denoted by $S^*O(X,\tau)$ (resp. $S^*C(X,\tau)$) [1], this family from a topology on X which is finer than τ [3] .The s*-closure of A , denoted by $s^*-cl(A)$ is the intersection of all s*-closed sets which contains A [3] . A subset A of a topological space (X,τ) is s*-closed iff $A = s^*-cl(A)$ [3]. Also, a function $f:(X,\tau) \to (Y,\tau^*)$ is called s*-irresolute if the inverse image of every s*-open subset of Y is an s*-open set in X [4]. Some times s*-closed set is also called $\hat{g}$-closed set [5,6] (resp. s*g-closed set [3,7]).

Throughout this paper (X,τ) and (Y,τ^*) (or simply X and Y) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned .

2. S*-Convergence Of Nets

2.1.Definition:
A subset A of a topological space X is called an s*-neighborhood of a point x in X if there exists an s*-open set U in X such that $x \in U \subseteq A$. The family of all s*-neighborhoods of a point $x \in X$ is denoted by $N_{s*}(x)$.

2.2.Remark:
Since every open set is an s*-open , then every neighborhood of x is an s*-neighborhood of x ,but the converse is not true in general . Consider the following example :-

Example:

Let X any infinite set with indiscrete topology and $x \in X$, then $\{x\}$ is an s*-neighborhood of x, since $x \in \{x\} \subseteq \{x\}$, where $\{x\}$ is an s*-open set in X, while $\{x\}$ is not neighborhood of x .

2.3.Theorem:

A function $f : X \to Y$ from a topological space X to a topological space Y is s*-irresolute iff for each $x \in X$ and each s*-neighborhood V of $f(x)$ in Y, there is an s*-neighborhood U of x in X such that $f(U) \subseteq V$.

Proof: $\Rightarrow$

Let $f : X \to Y$ be an s*-irresolute function and V be an s*-neighborhood of $f(x)$ in Y . To prove that , there is an s*-neighborhood U of x in X such that $f(U) \subseteq V$. Since f is an s*-irresolute then, $f^{-1}(V)$ is an s*-neighborhood of x in X .

Let $U = f^{-1}(V) \Rightarrow f(U) = f(f^{-1}(V)) \subseteq V \Rightarrow f(U) \subseteq V$.

Conversely,

To prove that $f : X \to Y$ is s*-irresolute . Let V be an s*-open set in Y . To prove that $f^{-1}(V)$ is an s*-open in X . Let $x \in f^{-1}(V) \Rightarrow$ $f(x) \in V \Rightarrow$ V is an s*-neighborhood of $f(x)$. By hypothesis there is an s*-neighborhood U_x of x such that $f(U_x) \subseteq V$.

$\Rightarrow U_x \subseteq f^{-1}(V), \forall\, x \in f^{-1}(V) \Rightarrow \exists$ an s*-open set W_x of x such that $W_x \subseteq U_x \subseteq f^{-1}(V), \forall\, x \in f^{-1}(V) \Rightarrow \bigcup_{x \in f^{-1}(V)} W_x \subseteq f^{-1}(V)$.

Since $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \{x\} \subseteq \bigcup_{x \in f^{-1}(V)} W_x \Rightarrow f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} W_x$.

$\Rightarrow f^{-1}(V)$ is an s*-open in Y , since its a union of s*-open sets .
Thus $f : X \to Y$ is an s*-irresolute function .

2.4.Definition:

Let $(x_d)_{d \in D}$ be a net in a topological space X . Then $(x_d)_{d \in D}$ s*-converges to $x \in X$(written $x_d \xrightarrow{s*} x$) iff for each s*-neighborhood U of x ,there is some $d_0 \in D$ such that $d \geq d_0$ implies

$x_d \in U$. Thus $x_d \xrightarrow{s*} x$ iff each s*-neighborhood of x contains a tail of $(x_d)_{d \in D}$. This is sometimes said $(x_d)_{d \in D}$ s*-converges to x iff it is eventually in every s*- neighborhood of x . The point x is called an s*-limit point of $(x_d)_{d \in D}$.

2.5.Definition:

Let $(x_d)_{d \in D}$ be a net in a topological space X .Then $(x_d)_{d \in D}$ is said to have $x \in X$ as an s*-cluster point (written $x_d \overset{s*}{\propto} x$) iff for each s*-neighborhood U of x and for each $d \in D$,there is some $d_0 \geq d$ such that $x_{d_0} \in U$. This is sometimes said $(x_d)_{d \in D}$ has x as an s*-cluster point iff $(x_d)_{d \in D}$ is frequently in every s*-neighborhood of x .

2.6.Theorem:

Let A be a subset of a topological space X . Then $x \in s*-cl(A)$ if and only if for any s*-open set U containing x , $A \cap U \neq \phi$.

Proof: $\Rightarrow$

Let $x \in s*-cl(A)$ and suppose that, there is an s*-open set U in X such that $x \in U$ & $A \cap U = \phi \Rightarrow A \subset U^c$ which is s*-closed in X $\Rightarrow s*-cl(A) \subseteq U^c$.

$\because x \in U \Rightarrow x \notin U^c \Rightarrow x \notin s*-cl(A)$, this is a contradiction .

Conversely,

Suppose that, for any s*-open set U containing x , $A \cap U \neq \phi$.
To prove that $x \in s*-cl(A)$,if not $\Rightarrow x \notin s*-cl(A)$

$\Rightarrow x \in (s^*-cl(A))^c$ which is s*-open in X $\Rightarrow A \bigcap (s^*-cl(A))^c \neq \phi$
.
This is a contradiction , since $A \bigcap (s^*-cl(A))^c = \phi$. Thus $x \in s^*-cl(A)$.

Since every neighborhood is an s*-neighborhood, then we have the following theorem:-

2.7.Theorem:
Let X be a topological space and $(x_d)_{d \in D}$ be a net in X and $x \in X$. Then :-

i) If $x_d \xrightarrow{s^*} x$, then $x_d \overset{s^*}{\propto} x$.

ii) If $x_d \xrightarrow{s^*} x$ $(x_d \overset{s^*}{\propto} x)$,then $x_d \to x$ $(x_d \propto x)$ respectively.

2.8.Remarks:

1) The converse of (2.7(i)) may not be true in general . To show that we give the following example:-

Example: Let $(\Re, \mu)$ be the usual topological space where $\Re$ be the set of all real numbers, then the net $(s_n)_{n \in N} = (n + (-1)^n n)_{n \in N}$ in $\Re$ has 0 as an s*-cluster point but not s*-limit point . Since if U is an s*-neighborhood of 0 in $\Re$, then for each $n \in N$, either n is odd or even .If n is odd, then $n_0 = n \Rightarrow s_{n_0} = 0 \in U$ and if n is even, then $n_0 = n+1 \Rightarrow s_{n_0} = 0 \in U$, thus $s_n \overset{s^*}{\propto} 0$. But s_n does not s*-converge to 0, since $U = (-1,1)$ is an s*-neighborhood of 0 and $s_n \notin (-1,1), \forall n \in N_e$.

2) The converse of (2.7(ii)) may not be true in general .To show that we give the following example:-

Example: Let (N, I) be the indiscrete topological space where N be the set of all natural numbers and $(s_n)_{n\in N} = (n)_{n\in N}$ be a net in N. Observe that $s_n \to 1$ ($s_n \propto 1$). But s_n does not s*-converge to 1 (does not s*-cluster to 1) , since $\{1\}$ is an s*-neighborhood of 1 and $s_n \notin \{1\}, \forall n > 1$.

2.9.Theorem:

Let X be a topological space and $A \subseteq X$. If x is a point of X , then $x \in s^*-cl(A)$ if and only if there exists a net $(x_d)_{d\in D}$ in A such that $x_d \xrightarrow{s^*} x$.

Proof: $\Leftarrow$

Suppose that $\exists$ a net $(x_d)_{d\in D}$ in A such that $x_d \xrightarrow{s^*} x$. To prove that $x \in s^*-cl(A)$. Let $U \in N_{s^*}(x)$, since $x_d \xrightarrow{s^*} x \Rightarrow \exists\, d_0 \in D$ such that $x_d \in U \;\forall\; d \geq d_0$.

But $x_d \in A \;\forall d \in D$. $\Rightarrow U \cap A \neq \phi \quad \forall\, U \in N_{s^*}(x)$. Hence by (2.6), we get $x \in s^*-cl(A)$.

Conversely,

Suppose that $x \in s^*-cl(A)$. To prove that $\exists$ a net $(x_d)_{d\in D}$ in A such that $x_d \xrightarrow{s^*} x$.

$\because\; x \in s^*-cl(A)$,then by (2.6), we get $N \cap A \neq \phi \;\; \forall\, N \in N_{s^*}(x)$.

$\therefore\; D = N_{s^*}(x)$ is a directed set by inclusion .

$\because\; N \cap A \neq \phi \;\; \forall\; N \in N_{s^*}(x) \;\Rightarrow\; \exists\, x_N \in N \cap A$.

Define $x : N_{s^*}(x) \to A$ by : $x(N) = x_N \;\; \forall\; N \in N_{s^*}(x)$.

$\therefore\; (x_N)_{N\in N_{s^*}(x)}$ is a net in A . To prove that $x_N \xrightarrow{s^*} x$.

Let $N \in N_{s^*}(x)$ to find $d_0 \in D$ such that $x_d \in N \;\; \forall\; d \geq d_0$.

Let $d_0 = N \Rightarrow \forall d \geq d_0 \;\Rightarrow\; d = M \in N_{s^*}(x)$.

i.e. $M \geq N \Leftrightarrow M \subseteq N$.

$\Rightarrow\; x_d = x(d) = x(M) = x_M \in M \cap A \subseteq M \subseteq N \Rightarrow\; x_M \in N$.

$\Rightarrow x_d \in N \ \forall d \geq d_0$. Thus $x_N \xrightarrow{s*} x$.

2.10.Definition:[4].

A topological space X is called an s*-T_2-space if for any two distinct points x and y of X,there are two s*-open sets U and V such that $x \in U, y \in V$ and $U \cap V = \phi$.

2.11.Theorem:

A topological space X is an s*-T_2-space iff every s*-convergent net in X has a unique s*-limit point .

Proof: $\Rightarrow$

Let X be an s*-T_2-space and $(x_d)_{d \in D}$ be a net in X such that $x_d \xrightarrow{s*} x$ & $x_d \xrightarrow{s*} y$ & $x \neq y$. Since X is an s*-T_2-space $\Rightarrow \exists U \in N_{s*}(x)$ and $V \in N_{s*}(y)$ such that $U \cap V = \phi$.

$\because x_d \xrightarrow{s*} x \Rightarrow \exists d_0 \in D$ $s.t$ $x_d \in U \ \forall d \geq d_0$.

$\because x_d \xrightarrow{s*} y \Rightarrow \exists d_1 \in D$ $s.t$ $x_d \in V \ \forall d \geq d_1$.

Since D is a directed set and $d_0, d_1 \in D$

$\Rightarrow \exists d_2 \in D$ $s.t$ $d_2 \geq d_0$ & $d_2 \geq d_1$.

$\Rightarrow x_d \in U \ \forall d \geq d_2$ and $x_d \in V \ \forall d \geq d_2 \Rightarrow U \cap V \neq \phi$.

This is a contradiction .

Conversely,

Suppose that every s*-convergent net in X has a unique s*-limit point . To prove that X is an s*-T_2-space . Suppose not $\Rightarrow \exists x, y \in X, x \neq y$ s.t $\forall U \in N_{s*}(x)$ and $\forall V \in N_{s*}(y), U \cap V \neq \phi$.

$\therefore (N_{s*}(x), \subseteq)$ and $(N_{s*}(y), \subseteq)$ are directed sets by inclusion .

Let $\rho = N_{s*}(x) \times N_{s*}(y)$. Define a relation $\geq$ on ρ as follows :

$\forall (U,V),(W,S) \in \rho$, we have $(U,V) \geq (W,S) \Leftrightarrow U \geq W$ & $V \geq S$.

It is easy to verify that $(\rho, \geq)$ is a directed set .

Let $(U,V) \in \rho \Rightarrow x \in U, y \in V$ & $U \cap V \neq \phi$.

$\because U \bigcap V \neq \phi \Rightarrow \exists\, x_{(U,V)} \in U \bigcap V$.

Define $x : \rho \to X$ by : $x(U,V) = x_{(U,V)} \ \forall (U,V) \in \rho$.

$\Rightarrow (x_{(U,V)})_{(U,V)\in\rho}$ is a net in X . We will show that $(x_{(U,V)})_{(U,V)\in\rho}$ is s*-convergent to both x and y .

For if $U \in N_{s*}(x)$ and $V \in N_{s*}(y)$, then for each $(N,M) \in \rho$ s.t $(N,M) \geq (U,V)$, we have $x(N,M) = x_{(N,M)} \in N \bigcap M \subseteq U \bigcap V$

$\Rightarrow x_{(N,M)} \in U$ and $x_{(N,M)} \in V$.

$\Rightarrow x_{(U,V)} \xrightarrow{s*} x$ and $x_{(U,V)} \xrightarrow{s*} y$.

This is a contradiction . Thus (X,τ) is an s*-T_2-space .

2.12.Definition:

Let X be a topological space and $A \subseteq X$. A point $x \in X$ is said to be s*-limit point of A iff every s*-open set U in X containing x contains a point of A different from x .

2.13.Theorem:

Let X be a topological space and $A \subseteq X$. Then:-

1. A point $x \in X$ is an s*-limit point of A iff there is a net $(x_d)_{d\in D}$ in $A - \{x\}$ s*-converging to x .
2. A set A is s*-closed in X iff no net in A s*-converges to a point in $X - A$.
3. A set A is s*-open in X iff no net in $X - A$ s*-converges to a point in A .

Proof:

1) $\Rightarrow$

Let x be an s*-limit point of A . To prove that $\exists$ a net $(x_d)_{d\in D}$ in $A - \{x\}$ such that $x_d \xrightarrow{s*} x$.

Since x is an s*-limit point of A $\Rightarrow \forall N \in N_{s*}(x)$, $N \bigcap A - \{x\} \neq \phi$.

$\therefore (N_{s*}(x), \subseteq)$ is a directed set by inclusion .

Since $N \cap A-\{x\} \neq \phi$, $\forall N \in N_{s*}(x) \Rightarrow \exists x_N \in N \cap A-\{x\}$.
Define $x : N_{s*}(x) \to A-\{x\}$ by : $x(N) = x_N \ \ \forall N \in N_{s*}(x)$.

$\therefore (x_N)_{N \in N_{s*}(x)}$ is a net in $A-\{x\}$. To prove that $x_N \xrightarrow{s*} x$.
Let $N \in N_{s*}(x)$ to find $d_0 \in D$ such that $x_d \in N \ \forall d \geq d_0$.
Let $d_0 = N \Rightarrow \forall d \geq d_0 \Rightarrow d = M \in N_{s*}(x)$.
i.e. $M \geq N \Leftrightarrow M \subseteq N$.
$\therefore x_d = x(d) = x(M) = x_M \in M \cap A-\{x\} \subseteq M \subseteq N \Rightarrow x_M \in N$.
$\Rightarrow x_d \in N \ \ \forall d \geq d_0$. Thus $x_N \xrightarrow{s*} x$.

Conversely,

Suppose that $\exists$ a net $(x_d)_{d \in D}$ in $A-\{x\}$ such that $x_d \xrightarrow{s*} x$.
To prove that x is an s*-limit point of A. Let $U \in N_{s*}(x)$, since $x_d \xrightarrow{s*} x \Rightarrow \exists d_0 \in D$ such that $x_d \in U \ \forall d \geq d_0$.
But $x_d \in A-\{x\} \ \forall d \in D \Rightarrow U \cap A-\{x\} \neq \phi \ \forall U \in N_{s*}(x)$.
Thus x is an s*-limit point of A .

2) $\Rightarrow$

Let A be an s*-closed in X . To prove that $\exists$ no net in A s*-converges to a point in $X-A$.

Suppose not $\Rightarrow \exists$ a net $(x_d)_{d \in D}$ in A s.t $x_d \xrightarrow{s*} x$ and $x \in X-A$.
By (2.9) $x \in s*-cl(A)$. Since A is s*-closed in X ,then $s*-cl(A) = A \Rightarrow x \in A$. But $x \in X-A \Rightarrow (X-A) \cap A \neq \phi$, this is a contradiction .
Thus no net in A s*-converges to a point in X − A .

Conversely,

Suppose that $\exists$ no net in A s*-converges to a point in X − A .
To prove that A is s*-closed .Let $x \in s*-cl(A)$,then by (2.9) $\exists$ a net $(x_d)_{d \in D}$ in A such that $x_d \xrightarrow{s*} x$. By hypothesis ,we get every net in A s*-converges to a point in A

$\Rightarrow x \in A \Rightarrow s^*\!-cl(A) \subseteq A$. Since $A \subseteq s^*\!-cl(A) \Rightarrow A = s^*\!-cl(A)$
$\Rightarrow$ A is s*-closed .

3) By (2) A is s*-open in X iff $X - A$ is s*-closed in X iff no net in $X - A$ s*-converges to a point in A .

2.14.Remarks: Let $(x_d)_{d\in D}$ be a net in a topological space X and $x \in X$. Then:-

1) If $x_d \xrightarrow{s^*} x$, then every subnet of $(x_d)_{d\in D}$ s*-converges to x .

2) If every subnet of $(x_d)_{d\in D}$ has a subnet s*-convergent to x , then $x_d \xrightarrow{s^*} x$.

3) If $x_d = x, \forall d \in D$, then $x_d \xrightarrow{s^*} x$.

2.15.Theorem:

Let X and Y be topological spaces . A function $f : X \to Y$ is an s*-irresolute iff whenever $(x_d)_{d\in D}$ is a net in X such that $x_d \xrightarrow{s^*} x$, then $f(x_d) \xrightarrow{s^*} f(x)$.

Proof: $\Rightarrow$

Suppose that $f : X \to Y$ is an s*-irresolute and $(x_d)_{d\in D}$ be a net in X s.t $x_d \xrightarrow{s^*} x$. To prove that $f(x_d) \xrightarrow{s^*} f(x)$.

Let $V \in N_{s^*}(f(x))$, since f is s*-irresolute, then by (2.3) $\exists\, U \in N_{s^*}(x)$ s.t $f(U) \subseteq V$. Since $U \in N_{s^*}(x)$ and $x_d \xrightarrow{s^*} x$.

$\Rightarrow \exists d_0 \in D$ s.t $x_d \in U, \forall d \ge d_0$.

$\Rightarrow \exists d_0 \in D$ s.t $f(x_d) \in f(U) \subseteq V$, $\forall\, d \ge d_0$.

$\therefore\ \forall\, V \in N_{s^*}(f(x)), \exists d_0 \in D$ s.t $f(x_d) \in V$, $\forall\, d \ge d_0$.

Thus $f(x_d) \xrightarrow{s^*} f(x)$.

Conversely,

To prove that $f: X \to Y$ is s*-irresolute . Suppose not, then by (2.3) $\exists V \in N_{s*}(f(x))$ s.t $\forall\, U \in N_{s*}(x), f(U) \not\subset V$.

$\therefore \forall U \in N_{s*}(x), \exists\, x_U \in U$ s.t $f(x_U) \notin V$.

$\therefore (N_{s*}(x), \subseteq)$ is a directed set by inclusion .

Define $x: N_{s*}(x) \to X$ by : $x(U) = x_U \quad \forall U \in N_{s*}(x)$.

$\therefore (x_U)_{U \in N_{s*}(x)}$ is a net in X . To prove that $x_U \xrightarrow{s*} x$.

Let $U \in N_{s*}(x)$ to find $d_0 \in D$ such that $x_d \in U \quad \forall\, d \geq d_0$.

Let $d_0 = U \Rightarrow \forall\, d \geq d_0 \Rightarrow d = N \in N_{s*}(x)$.

i.e. $N \geq U \Leftrightarrow N \subseteq U$.

$\therefore x(N) = x_N \in N \subseteq U \Rightarrow x_N \in U \;\forall d \geq d_0 \Rightarrow x_U \xrightarrow{s*} x$.

But $(f(x_U))$ does not s*-converges to f(x), since $f(x_U) \notin V \;\forall\, U \in N_{s*}(x)$. This is a contradiction . Thus $f: X \to Y$ is an s*-irresolute .

2.16.Theorem:

Let $(x_d)_{d \in D}$ be a net in a topological space X and for each d in D let A_d be the set of all points x_{d_0} for $d_0 \geq d$. Then x is an s*-cluster point of $(x_d)_{d \in D}$ if and only if x belongs to the s*-closure of A_d for each d in D .

Proof: $\Rightarrow$

If x is an s*-cluster point of $(x_d)_{d \in D}$, then for each d , A_d intersects each s*-neighborhood of x because $(x_d)_{d \in D}$ is frequently in each s*-neighborhood of x. Therefore x is in the s*-closure of each A_d.

Conversely,

If x is not an s*-cluster point of $(x_d)_{d \in D}$, then there is an s*-neighborhood U of x such that $(x_d)_{d \in D}$ is not frequently in U . Hence for some d in D , if $d_0 \geq d$, then $x_{d_0} \notin U$, so that U and A_d

are disjoint . Consequently x is not in the s*-closure of A_d.

3. S*-Convergence Of Filters

3.1.Definition:
A filter ξ on a topological space X is said to s*-converge to $x \in X$ (written $\xi \xrightarrow{s^*} x$) iff $N_{s^*}(x) \subseteq \xi$.

3.2.Definition:
A filter ξ on a topological space X has $x \in X$ as an s*-cluster point (written $\xi \overset{s^*}{\propto} x$) iff each $F \in \xi$ meets each $N \in N_{s^*}(x)$.

3.3.Remark:
A filter ξ on a topological space X has $x \in X$ as an s*-cluster point iff $x \in \bigcap\{s^*\text{-}cl(F) : F \in \xi\}$.

Proof: To prove that $\xi \overset{s^*}{\propto} x \Leftrightarrow x \in \bigcap\{s^*\text{-}cl(F) : F \in \xi\}$.

$$\begin{aligned}\therefore\ \xi \overset{s^*}{\propto} x &\Leftrightarrow \forall N \in N_{s^*}(x)\ \&\ \forall F \in \xi, N \cap F \neq \phi \\ &\Leftrightarrow \forall N \in N_{s^*}(x), F \cap N \neq \phi, \forall F \in \xi \\ &\Leftrightarrow x \in s^*\text{-}cl(F), \forall F \in \xi \\ &\Leftrightarrow x \in \bigcap\{s^*\text{-}cl(F) : F \in \xi\}.\end{aligned}$$

3.4.Theorem:
Let X be a topological space and ξ be a filter on X and $x \in X$. Then :-

1) If $\xi \xrightarrow{s^*} x$, then $\xi \overset{s^*}{\propto} x$.

2) If $\xi \xrightarrow{s^*} x$, then $\xi \to x$.

3) If $\xi \overset{s^*}{\propto} x$, then $\xi \propto x$.

4) If $\xi \xrightarrow{s^*} x$, then every filter finer than ξ also s*-converges to x.

Proof: It is a obvious .

3.5.Remark:
The converse of (3.4) may not be true in general . To show that we give the following examples:

Examples:

1) Let $(\Re, \mu)$ be the usual topological space where $\Re$ be the set of all real numbers and $\xi = \{A \subseteq \Re : [-1,1] \subseteq A\}$ be a filter on $\Re$, then $\xi \overset{s*}{\propto} 0$, but ξ does not s*-converge to 0 , since $(-1,1) \in N_{s*}(0)$, but $(-1,1) \notin \xi$.

2) Let $X = \{1,2,3\}$ & $\tau = \{\phi, X, \{1,2\}\}$
$\Rightarrow S^*O(X) = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$.
Let $\xi = \{X, \{1,2\}\}$ be a filter on X .
$\because N(1) = \{X, \{1,2\}\} \Rightarrow N(1) \subseteq \xi \Rightarrow \xi \to 1$.
$\because N_{s*}(1) = \{X, \{1\}, \{1,2\}, \{1,3\}\} \Rightarrow N_{s*}(1) \not\subset \xi$
$\Rightarrow \xi$ is not s*-converge to 1.

3) Let $X = \{1,2,3\}$ & $\tau = \{\phi, X\}$
$\Rightarrow S^*O(X) = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$.

Let $\xi = \{X, \{1,2\}\}$ be a filter on X .
$\because N(3) = \{X\} \Rightarrow \xi \propto 3$.
$\because N_{s*}(3) = \{X, \{3\}, \{2,3\}, \{1,3\}\} \Rightarrow \xi$ is not s*-cluster to 3 , since $\{3\} \cap \{1,2\} = \phi$.

4) Let $X = \{1,2\}$ & $\tau = \{\phi, X, \{1\}\} \Rightarrow S^*O(X) = \{\phi, X, \{1\}\}$.
Let $\xi' = \{X, \{1\}\}$ & $\xi = \{X\}$.
$\because N_{s*}(1) = \{X, \{1\}\} \Rightarrow N_{s*}(1) \subseteq \xi' \Rightarrow \xi' \xrightarrow{s*} 1$.

But $\xi \subseteq \xi'$ and ξ is not s*-converge to 1, since $N_{s^*}(1) \not\subset \xi$.

3.6.Definition:

A filter base ξ_0 on a topological space X is said to s*-converge to $x \in X$ (written $\xi_0 \xrightarrow{s^*} x$) iff the filter generated by ξ_0 s*-converges to x .

3.7.Definition:

A filter base ξ_0 on a topological space X has $x \in X$ as an s*-cluster point (written $\xi_0 \overset{s^*}{\propto} x$) iff each $F_0 \in \xi_0$ meets each $N \in N_{s^*}(x)$ (iff the filter generated by ξ_0 s*-clusters at x).

3.8.Theorem:

A filter base ξ_0 on a topological space X s*-converges to $x \in X$ iff for each $N \in N_{s^*}(x)$, there is $F_0 \in \xi_0$ such that $F_0 \subseteq N$.

Proof: $\Rightarrow$

Given $\xi_0 \xrightarrow{s^*} x$, then the filter ξ generated by ξ_0 s*-converges to x. i.e. $\xi \xrightarrow{s^*} x \Rightarrow N_{s^*}(x) \subseteq \xi \Rightarrow \forall N \in N_{s^*}(x), N \in \xi$

$\Rightarrow \exists\, F_0 \in \xi_0 \;\; s.t \;\; F_0 \subseteq N$.

Conversely,

To prove that $\xi_0 \xrightarrow{s^*} x$ .i.e. ξ generated by ξ_0 s*-converges to x. Let $N \in N_{s^*}(x)$,then by hypothesis, $\exists\, F_0 \in \xi_0 \;\; s.t \;\; F_0 \subseteq N$, since ξ is a filter, then $N \in \xi \Rightarrow N_{s^*}(x) \subseteq \xi \Rightarrow \xi \xrightarrow{s^*} x$

$\Rightarrow \xi_0 \xrightarrow{s^*} x$.

3.9.Theorem:

A filter ξ on a topological space X has $x \in X$ as an s*-cluster point iff there is a filter ξ' finer than ξ which s*-converges to x.

Proof: $\Rightarrow$

If $\xi \overset{s^*}{\propto} x$, then by (3.2) each $F \in \xi$ meets each $N \in N_{s^*}(x)$.

$\Rightarrow$ $\xi'_0 = \{N \bigcap F : N \in N_{s^*}(x), F \in \xi\}$ is a filter base for some filter ξ' which is finer than ξ and s*-converges to x.

Conversely,

Given $\xi \subseteq \xi'$ and $\xi' \xrightarrow{s^*} x \Rightarrow$ $\xi \subseteq \xi'$ and $N_{s^*}(x) \subseteq \xi'$.

$\Rightarrow$ each $F \in \xi$ and each $N \in N_{s^*}(x)$ belong to ξ'.

Since ξ' is a filter $\Rightarrow$ $N \bigcap F \neq \phi$ $\Rightarrow \xi \overset{s^*}{\propto} x$.

3.10.Theorem:

Let X be a topological space and $A \subseteq X$. Then $x \in s^*\!-\!cl(A)$ iff there is a filter ξ such that $A \in \xi$ and $\xi \xrightarrow{s^*} x$.

Proof: $\Rightarrow$

If $x \in s^*\!-\!cl(A) \Rightarrow U \bigcap A \neq \phi$ $\forall U \in N_{s^*}(x)$.

$\Rightarrow \xi_0 = \{U \bigcap A : U \in N_{s^*}(x)\}$ is a filter base for some filter ξ.

The resulting filter contains A and $\xi \xrightarrow{s^*} x$.

Conversely,

If $A \in \xi$ and $\xi \xrightarrow{s^*} x \Rightarrow A \in \xi$ and $N_{s^*}(x) \subseteq \xi$.

Since ξ is a filter $\Rightarrow U \bigcap A \neq \phi$ $\forall U \in N_{s^*}(x) \Rightarrow x \in s^*\!-\!cl(A)$.

3.11.Definition[8]:

Let X and Y be topological spaces, $f : X \to Y$ be a function and ξ be a filter on X, then $f(\xi)$ is the filter on Y having for a base the sets $\{ f(F) : F \in \xi \}$.

3.12.Theorem:

Let X and Y be two topological spaces . A function

$f: X \to Y$ is an s*-irresolute iff whenever $\xi \xrightarrow{s^*} x$ in X, then $f(\xi) \xrightarrow{s^*} f(x)$ in Y .

Proof: $\Rightarrow$

Suppose that $f: X \to Y$ is s*-irresolute and $\xi \xrightarrow{s^*} x$.
To prove that $f(\xi) \xrightarrow{s^*} f(x)$ in Y . Let $V \in N_{s^*}(f(x))$, since f is s*-irresolute, then by (2.3),there is an s*-neighborhood U of x such that $f(U) \subseteq V$. Since $\xi \xrightarrow{s^*} x$, then $U \in \xi \Rightarrow f(U) \in f(\xi)$.
But $f(U) \subseteq V$,then $V \in f(\xi)$. Thus $f(\xi) \xrightarrow{s^*} f(x)$.

Conversely,

Suppose that whenever $\xi \xrightarrow{s^*} x$ in X, then $f(\xi) \xrightarrow{s^*} f(x)$ in Y.
To prove that $f: X \to Y$ is s*-irresolute .
Let $\xi = \{U : U \in N_{s^*}(x)\} \Rightarrow \xi$ is a filter on X and $\xi \xrightarrow{s^*} x$.
By hypothesis $f(\xi) \xrightarrow{s^*} f(x) \Rightarrow$ each $V \in N_{s^*}(f(x))$ belongs to $f(\xi)$
$\Rightarrow \exists U \in N_{s^*}(x)$ $s.t$ $f(U) \subseteq V \Rightarrow$ $f: X \to Y$ is an s*-irresolute function .

3.13.Theorem:

Let X be a topological space and $A \subseteq X$. Then a point $x \in X$ is an s*-limit point of A iff $A - \{x\}$ belongs to some filter which s*-converges to x.

Proof: $\Rightarrow$

If x is an s*-limit point of $A \Rightarrow U \cap A - \{x\} \neq \phi$ $\forall U \in N_{s^*}(x)$.
$\Rightarrow \xi_0 = \{U \cap A - \{x\} : U \in N_{s^*}(x)\}$ is a filter base for some filter ξ.
The resulting filter contains $A - \{x\}$ and $\xi \xrightarrow{s^*} x$.

Conversely,

If $A-\{x\}\in\xi$ and $\xi\xrightarrow{s^*}x \Rightarrow A-\{x\}\in\xi$ and $N_{s^*}(x)\subseteq\xi$. Since ξ is a filter $\Rightarrow U\bigcap A-\{x\}\neq\phi\ \ \forall U\in N_{s^*}(x)$. Thus x is an s*-limit point of a set A.

3.14.Definition[8]:
If $(x_d)_{d\in D}$ is a net in a topological space X, the filter generated by the filter base ξ_0 consisting of the sets $B_{d_0}=\{x_d : d\geq d_0\}, d_0\in D$ is called the filter generated by $(x_d)_{d\in D}$.

3.15.Theorem:
A net $(x_d)_{d\in D}$ in a topological space X s*-converges to $x\in X$ iff the filter generated by $(x_d)_{d\in D}$ s*-converges to x.

Proof: The net $(x_d)_{d\in D}$ s*-converges to x iff each s*-neighborhood of x contains a tail of $(x_d)_{d\in D}$, since the tails of $(x_d)_{d\in D}$ are a base for the filter generated by $(x_d)_{d\in D}$, the result follows.

3.16.Definition[8]:
If ξ is a filter on a topological space X, let $D_\xi=\{(x,F): x\in F\in\xi\}$. Then D_ξ is directed by the relation $(x_1,F_1)\leq(x_2,F_2)$ iff $F_2\subseteq F_1$, so the function $p: D_\xi\to X$ defined by $p(x,F)=x$ is a net in X. It is called the net based on ξ.

3.17.Theorem:
A filter ξ on a topological space X s*-converges to $x\in X$ iff the net based on ξ s*-converges to x.

Proof: $\Rightarrow$

Suppose that $\xi \xrightarrow{s^*} x$. If $N \in N_{s^*}(x)$, then $N \in \xi$. Since $N \neq \phi$, then $\exists\ p \in N$. Let $d_0 = (p, N) \in D_\xi$.

$\therefore \forall d = (q, F) \geq d_0 = (p, N) \Rightarrow x_d = x_{(q,F)} = q \in F \subseteq N.$

Thus the net based on ξ s*-converges to x.

Conversely,

suppose that the net based on ξ s*-converges to x. Let $N \in N_{s^*}(x)$, then $\exists d_0 = (p_0, F_0) \in D_\xi$ such that $x_{(p,F)} = p \in N, \forall (p, F) \geq (p_0, F_0)$. Then $F_0 \subseteq N$; otherwise, there is some $q \in F_0 - N$, and then $(q, F_0) \geq (p_0, F_0)$, but $x_{(q,F_0)} = q \notin N$. Hence $N \in \xi$, so $\xi \xrightarrow{s^*} x$.

3.18.Theorem:

A topological space X is an s*-T_2-space iff every s*-convergent filter in X has a unique s*-limit point .

Proof: $\Rightarrow$

Let X be an s*-T_2-space and ξ be a filter in X such that $\xi \xrightarrow{s^*} x$ & $\xi \xrightarrow{s^*} y$ & $x \neq y$. Since X is an s*-T_2-space $\Rightarrow \exists U \in N_{s^*}(x)$ and $V \in N_{s^*}(y)$ such that $U \cap V = \phi$.

$\because\ \xi \xrightarrow{s^*} x \Rightarrow N_{s^*}(x) \subseteq \xi$.

$\because\ \xi \xrightarrow{s^*} y \Rightarrow N_{s^*}(y) \subseteq \xi$.

$\because\ U \in N_{s^*}(x) \subseteq \xi$ & $V \in N_{s^*}(y) \subseteq \xi \Rightarrow U, V \in \xi$.

Since ξ is a filter , then $U \cap V \neq \phi$. This is a contradiction. Hence ξ s*-converges to a unique s*-limit point.

Conversely,

To prove that X is an s*-T_2-space . Suppose not, then $\exists\ x, y \in X, x \neq y$ s.t $\forall U \in N_{s^*}(x)$ & $\forall V \in N_{s^*}(y)$, $U \cap V \neq \phi$.

$\Rightarrow \ \xi_0 = \{U \cap V : U \in N_{s*}(x), V \in N_{s*}(y)\}$ is a filter base for some filter ξ. The resulting filter s*-converges to x and y.
This is a contradiction . Thus X is an s*-T_2-space .

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حول تقارب الشبكات والمرشحات - *S

صبيحة إبراهيم محمود

الجامعة المستنصرية / كلية العلوم / قسم الرياضيات

المستخلص

كرس هذا البحث لتقديم و دراسة العديد من الخواص التبولوجية لتقارب الشبكات من النمط- *s (s* - convergence of nets) وتقارب المرشحات من النمط - *s (s* - convergence of filters) مستخدمين مفهوم المجموعات المفتوحة من النمط - *s (s* -open sets) . كذلك درسنا بعض خواص النقاط العنقودية من النمط –*s (s* -cluster points) للشبكات والمرشحات.