On The Range of Generalized Jordan *-derivation

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Abstract : *Let H be an infinite dimensional separable complex Hilbert space and let B(H) be the algebra of bounded liners operators in H.*

Let $A,B \in B(H)$, *the generalized Jordan *-derivations* $J_{A,B} : B(H) \longrightarrow B(H)$ *is defined by:*

$$J_{A,B}(X) = XA - BX^*, \qquad X \in B(H)$$

*In this paper we study some properties of the range of generalized Jordan *-derivations, which is represented by* $R_{A,B}$, *and defined by:*

$$R_{A,B} = \{XA - BX^* : X \in B(H)\}.$$

Introduction

Let R be a *-ring. An additive mapping $J:R\longrightarrow R$ is called a Jordan *-derivation if it satisfies

$$J(a^2)=aJ(a)+J(a)a^* \qquad (a\in R)$$

Jordan *-derivations were introduced in the papers of Semrl ([7], [8]). The reason for introducing these mapping was the fact that the problem of representing quadratic forms by sesqulinear ones is closely connected with the structure of Jordan *-derivations. In this subject we would refer to articles [9], [11].

The structure of Jordan *-derivations on standard operator algebras was described by Semrl [10]. From this his results it also follows that a Jordan *-derivation on the algebra B(H) of all bounded linear operators in real or complex Hilbert space H of the form

$$J(X)=TX-XT^* \qquad (X\in B(H))$$

For some operator $A\in B(H)$.

In [6] Pέter treated a question in regard to generalized Jordan *-derivation was the concept of Jordan -*derivation pairs, which was introduced by Zalar [11] given a good results with respect to the problem of representing quadratic forms by sesqulinlear ones on modules over *-ring.

Molnar has shown [5] that on standard operator algebras Jordan *-derivation pairs are of the form

$$E(X)=XA-BX^* \qquad F(X)=XB-AX^* \qquad (X\in B(H))$$

For some operator $A, B\in B(H)$.

In this paper we prove that under some conditions on the range of generalized Jordan *-derivation, A is a θ-adjoint invertible operator.

Also, we study the range of generalized Jordan *-derivation when A,B are hyponormal operators.

1. Preliminaries

We begin by the following definitions and propositions which will be used for the sequel.

Definition 1.1[2]

An operator A is called normal if $A^*A=AA^*$, equivalently, $\|Af\| = \|A^*f\|$ for each f∈H.

Remark 1.2 [3]

An element λ∈σ(A) is called an eigenvalue of A (λ∈$\prod_o$(A)) if there exists a non-zero vector x∈H such that Ax=λx. And λ is called an approximate eigenvalue of A (λ∈∏(A)) if there exists a sequence $\{u_n\}$ of unit vectors in H such that $\|(A-\lambda I)u_n\| \to 0$.

Equivalently, for each ε > 0 there exists a unit vector x∈H such that $\|Ax-\lambda x\| < \varepsilon$. It is known that if A is a normal operator, then every element in the spectrum is an approximate eigenvalue.

Definition 1.3[2]

An operator A is called hyponormal if $A^*A \geq AA^*$, equivalently $\|A^*f\| \leq \|Af\|$ for each f∈H.

Remark 1.4 [3]

Each $\lambda\in\partial\sigma(A)$ is an approximate eigenvalue. (Where $\partial\sigma(A)$ is the set of boundary points of the spectrum).

Definition 1.5 [4]

An operator A in B(H) is θ-adjoint if $A^*=e^{i\theta}A$, where $\theta\in R$.

Remarks 1.6 [4]

(1) If θ=0 then clearly A is θ-adjoint.

(2) If θ=π then A is θ-adjoint iff A is skew-Hermitian.

(3) Let A be a θ-adjoint operator, it is easy to verify that A is normal.

(4) If A and B are θ-adjoint operators then A+B and rA for each $r\in R$ are also θ-adjoint operators.

(5) If A is a normal operator, then A is θ-adjoint iff $\sigma(A)\subseteq e^{\frac{-i\theta}{2}}R$.

(6) If A is a hyponormal operator, then A is θ-adjoint iff $\sigma(A)\subseteq e^{\frac{-i\theta}{2}}R$.

2. The range of generalize Jordan*-derivation and θ-adjoint operator.

In this section we study the range of generalized Jordan*-derivation ($R_{A,B}$) where the operator $ire^{\frac{-i\theta}{2}}I\epsilon R_{A,B}$

Suppose that A,B are two normal operators. The following our main result shows that, if $ire^{\frac{-i\theta}{2}}I\epsilon R_{A,B}$, then A and B are two θ-adjoint invertible operators.

Theorem 2.1

Let A and B are two normal operators such that σ(A)=σ(B), and let θ∈R. If $ire^{\frac{-i\theta}{2}}I\epsilon R_{A,B}$ for some 0≠r∈R then A and B are θ-adjoint invertible operators.

Proof

Let ε>0, λ∈σ(A)∩σ(B), since λ is an approximate eigenvalue for A and B, then there exists a unit vector f in H such that

$\|Af - \lambda f\| < \frac{\varepsilon}{4}$ ………………………………………………(2.1)

$\|Bf - \lambda f\| < \frac{\varepsilon}{4}$ ……………………………………………… (2.2)

Since $ire^{\frac{-i\theta}{2}}I\epsilon R_{A,B}$, then there exists X∈B(H) such that $ire^{\frac{-i\theta}{2}}I - (XA - BX^*) = 0$

So,

$< (ire^{\frac{-i\theta}{2}}I - XA + BX^*)g, g >= 0$ for all g∈H.

In particular,

$$\left| < ire^{\frac{-i\theta}{2}}If, f > - < XAf, f > + < BX^*f, f > \right| = 0$$

Or,

$$\left|< ire^{\frac{-i\theta}{2}}If, f > - < Af, X^*f > + < X^*f, B^*f >\right| = 0 \qquad \text{………..(2.3)}$$

Note that since ε is arbitrary, then we can assume in equation (2.1) that

$$\|Af - \lambda f\|\|X^*\| < \frac{\varepsilon}{4}$$

By Schwarz inequality we get

$$|< Af, X^*f > -\lambda < f, X^*f >| < \tfrac{\varepsilon}{4} \qquad \text{………………………….(2.4)}$$

Now, the normality of B implies

$\|Bg\| = \|B^*g\|$ for all $g \in H$

Hence from equation (2.2)

$$\|X^*\|\left\|B^*f - \bar{\lambda}f\right\| < \frac{\varepsilon}{4}$$

Again by Schwarz inequality we get

$$\left|< X^*f, B^*f > - < X^*f, \bar{\lambda}f >\right| < \tfrac{\varepsilon}{4}$$

Or

$$|\lambda < X^*f, f > - < X^*f, B^*f >| < \tfrac{\varepsilon}{3} \qquad \text{………………………….(2.5)}$$

By adding the equations (2.3), (2.4) and (2.5)

$$\left|ire^{\frac{-i\theta}{2}}I - \lambda < f, X^*f > + \lambda < X^*f, f >\right| < \varepsilon$$

Or,

$$\left|ire^{\frac{-i\theta}{2}}I - \lambda < (X - X^*)f, f >\right| < \varepsilon$$

Since $(X - X^*)$ is skew-Hermitian operator, then it is easily seen that $< (X - X^*)f, f >$ is pure imaginary number say ic, c∈R.

Since ε is arbitrary, then $ire^{\frac{-i\theta}{2}} I - \lambda ic = 0$, that is $ire^{\frac{-i\theta}{2}} I = \lambda ic.$

Note that, λ≠0, because by assumption, r≠0.

Thus, λ=k$e^{\frac{-i\theta}{2}} \in e^{\frac{-i\theta}{2}} R$

Consequently, σ(A) and σ(B) ⊆ $e^{\frac{-i\theta}{2}} R$, thus by (1.6) A and B are θ-adjoint operators.

Since λ≠0 for all λ∈σ(A)=σ(B), then A and B are invertible.

As a special case of (theorem 2.1) we have

Remark 2.2

Let A and B are two normal operators such that σ(A)=σ(B).

1) If $irI \epsilon R_{A,B}$ for some 0≠r∈R, then A is invertible Hermitian operator.
2) If $rI \epsilon R_{A,B}$ for some 0≠r∈R, then A is invertible skew-Hermitian operator.

Proposition 2.3

Let A and B are two normal operators such that σ(A)=σ(B). If there exists K∈$R_{A,B}$ such that 0∉$\overline{W}(K)$, (where $\overline{W}(K)$ is the closure of the numerical range of K), then A and B are invertible.

Proof

Repeat the argument used in the proof of last theorem, we get the following inequalities:

$|< Kf, f > - < Af, X^*f > + < X^*f, B^*f >|=0$

$|< Af, X^*f > -\lambda < f, X^*f >| < \frac{\varepsilon}{4}$

$|\lambda < X^*f, f > -< X^*f, B^*f >| < \frac{\varepsilon}{4}$

Where $\lambda\in\sigma(A)=\sigma(B)$ and X, f satisfy

$K - (XA - BX^*) = 0$, $\|X^*\|\|B^*f - \bar{\lambda}f\| < \frac{\varepsilon}{4}$

and $\|Af - \lambda f\|\|X^*\| < \frac{\varepsilon}{4}$

Now, adding the inequalities above, we have
$|K - \lambda < (X - X^*)f, f >| < \varepsilon$

If $\lambda=0$ then $|< Bf, f >| < \varepsilon$, which means that $0\in\bar{W}(K)$, a contradiction. Consequently, $0\notin\sigma(A)=\sigma(B)$ and hence A, B are invertible.

The next theorem is generalization of theorem (2.1). Before we state this theorem we need the following:

Lemma (2.4) [4]

Let S be a convex set in the complex plane C, and let $A\in B(H)$. If $\partial\sigma(A)\subseteq S$ then $\sigma(A)\subseteq S$.

Now, suppose that A,B are two hyponormal operators. The following theorem shows that, if $ire^{\frac{-i\theta}{2}}I\epsilon R_{A,B}$, then A and B are two θ-adjoint invertible operators.

Theorem (2.5)

Let A and B are two hyponormal operators such that σ(A)=σ(B), and let θ∈R. If $ire^{\frac{-i\theta}{2}}I\epsilon R_{A,B}$ for some 0≠r∈R then A and B are θ-adjoint invertible operators.

Proof

Let ε>0, λ∈ ∂σ(A)= ∂σ(B), thus by remark (1.4) λ is an approximate eigenvalue for A and B, then there exists a unit vector *f* in H such that

$$\|Af - \lambda f\| < \frac{\varepsilon}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1)$$

$$\|Bf - \lambda f\| < \frac{\varepsilon}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2)$$

Since $ire^{\frac{-i\theta}{2}}I\epsilon R_{A,B}$, then there exists X∈B(H) such that $ire^{\frac{-i\theta}{2}}I - (XA - BX^*) = 0$

The hyponormality of B implies that

$\|B^*g\| \leq \|Bg\|$ for all g∈H

Apply the argument used in the proof of Theorem (2.1) to get $\partial\sigma(A) \subseteq e^{\frac{-i\theta}{2}}R\backslash\{0\}$ which implies, by Lemma (2.4), that $\sigma(A) \subseteq e^{\frac{-i\theta}{2}}R$ and $\sigma(B) \subseteq e^{\frac{-i\theta}{2}}R$. Note that $e^{\frac{-i\theta}{2}}R$ is convex set. Therefore A and B are θ-adjoint operators (2.4). It is easy observing that $\sigma(A) \subseteq e^{\frac{-i\theta}{2}}R$ implies $\sigma(A) \subseteq \partial\sigma(A)$. Consequently, $0 \notin \sigma(A)$ and so A is invertible.

The same as about the operator B, so B is invertible.

A generalization of (2.3) is given as follows

Proposition (2.6)

Let A and B are two hyponormal operators such that $\sigma(A)=\sigma(B)$. If there exists $K\in R_{A,B}$ such that $0\notin\overline{W}(K)$, (where $\overline{W}(K)$ is the closure of the numerical range of K), then $0\notin\partial\sigma(A)$ and A and $0\notin\partial\sigma(B)$.

3. The range of generalize Jordan*-derivation and hyponormal operator.

In this section In this we study the range of generalize Jordan*-derivation ($R_{A,B}$) where A or B is hyponormal operator. Before we state our main result in this section we need the following:

Lemma (3.1) [1]

Let $T\in B(H)$, and let p be any polynomial. If λ is any approximate eigenvalue for T, then $p(\lambda)$ is an approximate eigenvalue for $p(T)$.

Now, Suppose that, B is hyponormal operator. The following theorem shows that if $p(A)\in R_{A,B}$ and if $\lambda\in\sigma(A)\cap\sigma(B)$ such that λ is an approximate eigenvalue of A and B with the same corresponding approximate eigenvector, $\lambda\neq 0$, in general $\frac{p(\lambda)}{\lambda}$ is pure imaginary number.

Theorem (3.2)

Let $A, B\in B(H)$ such that B is hyponormal operator, let p be a polynomial in one variable.

If $p(A)\in R_{A,B}$, then for each $\lambda\in\sigma(A)\cap\sigma(B)$ such that λ is an approximate eigenvalue of A and B with the same corresponding approximate eigenvector, if $\lambda=0$ then $p(\lambda)=0$, in general $\frac{p(\lambda)}{\lambda}$ is pure imaginary.

Proof:

Let $\varepsilon>0$ and let $\lambda\in\sigma(A)\cap\sigma(B)$. Suppose that X=0, then $p(A)= XA-BX^*=0$. By the spectral mapping theorem $p(\sigma(A))=\sigma(p(A))=\sigma(0)=\{0\}$. It follows that $p(\lambda)=0$. Hence, if $\lambda=0$, then $p(\lambda)=0$, and otherwise $\frac{p(\lambda)}{\lambda}\in iR$.

Suppose that $X\neq0$. Since λ is an approximate eigenvalue for A and B with the same corresponding approximate eigenvector, then there exists a unit vector f in H such that:

$\|Af-\lambda f\|<\frac{\varepsilon}{4}$ …………………………………………. (3.1)

$\|Bf-\lambda f\|<\frac{\varepsilon}{4}$ …………………………………………. (3.2)

Since λ is an approximate eigenvalue for A then by (3.1), $p(\lambda)$ is an approximate eigenvalue for p(A). Thus

$$\|p(A)f-p(\lambda)f\|<\frac{\varepsilon}{4}$$

Hence $\|p(A)f-p(\lambda)f\|\|f\|<\frac{\varepsilon}{4}$

So, by Schwarz inequality

$|<p(A)f,f>-p(\lambda)|<\frac{\varepsilon}{4}$ ……………………………… (3.3)

And since $p(A)\in R_{A,B}$, then, there exists $X\in B(H)$ such that

$P(A)-(XA-BX^*)=0$

In particular $<(p(A)-(XA-BX^*))f,f>=0$

So, $|< p(A)f, f > - < XAf, f > + < BX^{*}f, f >| = 0$

Or,

$|< p(A)f, f > - < Af, X^{*}f > + < X^{*}f, B^{*}f >| = 0$ ……… (3.4)

Note that, since ε is arbitrary, then we can assume in equation (3.1) that

$\|Af - \lambda f\|\|X^{*}\| < \frac{\varepsilon}{4}$

By Schwarz inequality we get:

$|< Af, X^{*}f > - \lambda < f, X^{*}f >| < \frac{\varepsilon}{4}$ ………………………… (3.5)

Now, the hyponormality of B implies that

$\|B^{*} g\| \leq \|Bg\|$ for all $g \in H$

Hence,

$$\|X^{*}\|\left\|B^{*}f - \bar{\lambda}f\right\| < \frac{\varepsilon}{4}$$

Again by Schwarz inequality we get

$$\left|< X^{*}f, B^{*}f > - < X^{*}f, \bar{\lambda}f >\right| < \frac{\varepsilon}{4}$$

Or,

$|\lambda < X^{*}f, f > - < X^{*}f, B^{*}f >| < \frac{\varepsilon}{4}$ ………………………(3.6)

By adding the equations (3.3), (3.4), (3.5) and (3.6)

$|p(\lambda) - \lambda < Xf, f > + \lambda < X^{*}f, f >| < \varepsilon$

Or,

$|p(\lambda) - \lambda < (X - X^{*})f, f >| < \varepsilon$

Thus, we have $|p(\lambda)-\lambda ic| < \varepsilon \quad for\ all\ \varepsilon > 0$

Therefore, $p(\lambda)-\lambda ic=0\ , c \in R$

Hence, $\frac{p(\lambda)}{\lambda} \in iR$

Proposition (3.3)

Let A,B∈B(H), such that A is hyponormal operator, let p be a polynomial in one variable.

If p(A)∈R_{A^*,B^*}, then for each λ∈σ(A)∩σ(B) such that λ is an approximate eigenvalue of A and B with the same corresponding approximate eigenvector, if λ=0 then p(λ)=0, in general λp(λ) is pure imaginary number.

Proof:

Let ε>0, and let λ∈σ(A)∩σ(B). As in theorem (3.2) we can assume X≠0. Thus apply the same argument as in (3.2), we get:

$\|Af - \lambda f\| < \frac{\varepsilon}{4}$ ………………………………………………… (3.1)

$\|Bf - \lambda f\| < \frac{\varepsilon}{4}$ ……………………………………………….. (3.2)

$|< p(A)f, f > -p(\lambda)| < \frac{\varepsilon}{4}$ ……………………………..………(3.3)

Since p(A)∈R_{A^*,B^*}, then there exists X∈B(H) such that

$|< p(A)f, f > -< A^*f, X^*f > +< X^*f, Bf >| = 0$……….. (3.4)

The hyponormality of A implies

$\left|< A^*f, X^*f > -\bar{\lambda} < Xf, f >\right| < \frac{\varepsilon}{4}$ ………………………… (3.5)

Form 2 we get

$$\left|\bar{\lambda} < X^* f, f > - < X^* f, Bf >\right| < \frac{\varepsilon}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.6)}$$

By adding the equations (3.3), (3.4), (3.5) and (3.6)

$$\left|p(\lambda) - \bar{\lambda} < Xf, f > + \bar{\lambda} < X^* f, f >\right| < \varepsilon$$

Or,

$$\left|p(\lambda) - \bar{\lambda} < (X - X^*) f, f >\right| < \varepsilon$$

Thus, we have $\left|p(\lambda) - \bar{\lambda} ic\right| < \varepsilon \quad for\ all\ \varepsilon > 0$

Therefore, $p(\lambda) - \bar{\lambda} ic = 0,\ c \in R$

Hence, $\lambda p(\lambda) \in iR$.

In a similar way we prove the following:

Remark (3.4)

(1) Let A, B∈B(H) such that B is hyponormal operator, let p be a polynomial in one variable.

If p(B)∈$R_{A,B}$, then for each λ∈σ(A)∩σ(B) such that λ is an approximate eigenvalue of A and B with the same corresponding approximate eigenvector, if λ=0 then p(λ)=0, in general $\frac{p(\lambda)}{\lambda}$ is pure imaginary.

(2) Let A,B∈B(H), such that A is hyponormal operator, let p be a polynomial in one variable.

If p(B)∈R_{A^*,B^*}, then for each λ∈σ(A)∩σ(B) such that λ is an approximate eigenvalue of A and B with the same corresponding approximate eigenvector, if $\lambda=0$ then $p(\lambda)=0$, in general $\lambda p(\lambda)$ is pure imaginary number.

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حول مدى الدوال الاشتقاقية من النمط جوردان*- العامة

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المستخلص

ليكن H فضاء هلبرت وليكن B(H) جبر المؤثرات الخطية المقيدة على H. يقال لتطبيق ما $J:B(H)\longrightarrow B(H)$ انه تطبيق جوردان* الاشتقاقي الداخلي إذا تحقق ما يلي:

$$J(X)\equiv J_A(X)=XA-AX^*, \qquad X\in B(H)$$

حيث أن A مؤثر ما على H.

لقد شغل هذا التطبيق عددا من الباحثين منذ بداية التسعينات من القرن الماضي لارتباطه بمسألة تاريخية مهمة تخص مسألة تمثيل الصيغة التربيعية بصيغة شبه خطية وقد استطاع P. Šemrl صاحب هذا التعريف من إيجاد علاقة واضحة بين هذه المسألة وبين تركيب هذا التطبيق.

في الآونة الأخيرة ظهرت دراسة لتطبيق مهم اخر الذي يعتبر تعميم لتطبيق جوردان* الاشتقاقي الداخلي وهو

$$J(X)\equiv J_{A,B}=XA-BX^*, \qquad X\in B(H)$$

لقد قام عدد من الباحثين بدراسة خواص هذا التطبيق واهميته، ومن ضمنهم L. Molnar الذي اهتم بدراسة مدى هذا التطبيق. ندرس في هذا البحث مدى هذا التطبيق ونكمل جهود الباحثين في الحصول على نتائج جديدة، فقد درسنا المدى عندما يكون كل من A,B مؤثرا سويا ومؤثرا سويا ملتويا، فوجدنا كيف ان وجود عناصر معينة في مدى هذا التطبيق يؤثر على المؤثر A والمؤثر B.