On Fuzzy Conjugate Spaces of Fuzzy 2-Normed Spaces

Dr. Faria Ali C.

Al-Mustansiriyah University - College of Science

Department of Mathematics

Baghdad, Iraq.

Fiaria_altaai@yahoo.com.

Abstract: *In this paper we give the definition of fuzzy 2-bounded linear functional and the notions of his fuzzy norm is introduced. Also, we introduce fuzzy conjugate space of a fuzzy 2-normed linear space and give some facts that are related with it.*

Keywords: fuzzy 2-bounded linear functional, fuzzy conjugate space.

1. Introduction

The theory of 2-norm on a linear space has been introduced by Gahler in [1]. In [2], Bag and Samanta introduced a fuzzy norm on a linear space. The notation of a fuzzy 2-normed space is introduced in [3]. In this paper we prove conjugate space of 2-normed space is complete and based on the ideas that appeared in [4], we introduce the definitions of fuzzy 2-bounded linear functional on fuzzy 2-normed space and fuzzy conjugate space of fuzzy 2-normed space. Thereafter we give some theorems which is useful to prove the completeness of fuzzy conjugate space of fuzzy 2-normed space.

2. Conjugate Space of 2-Normed Space

In this section we give some basic concepts in a 2-normed space and then study the completeness of the conjugate space.

Definition (2.1), [3]:

Let X to be a linear space over F (where F is the field of real or complex numbers). A function $\|.,.\|: X^2 \longrightarrow R$, satisfy the following axioms:

(N_1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent.

(N_2) $\|x_1, x_2\| = \|x_2, x_1\|$

(N_3) $\|x_1, cx_2\| = |c|\|x_1, x_2\|$ for any $c \in F$.

(N_4) $\|x, y+z\| \le \|x, y\| + \|x, z\|$

is said to be a 2-norm on X and the pair $(X, \|.,.\|)$ is called a 2-normed space.

Definition (2.2), [5]:

Let $(X, \|.,.\|)$ be a 2-normed space. The linear functional T on X^2 is said to be 2-bounded on X^2 in case there exists M>0 such that

$|T(x_1, x_2)| \le M\|x_1, x_2\|$, for each $(x_1, x_2) \in X^2$

Proposition (2.3), [5]:

Let T be 2-bounded linear functional on a 2-normed linear space $(X, \|.,.\|)$ then

$$\|T\| = \sup\left\{\frac{|T(x_1, x_2)|}{\|x_1, x_2\|} : (x_1, x_2) \in X^2 \text{ and } \|x_1, x_2\| \neq 0\right\}$$

is a norm on the linear space of all 2-bounded linear functional defined on X^2.

Definition (2.4), [5]:

Let $(X,\|.,.\|)$ be a 2-normed linear space. We denote by $B(X^2,F)$ the set of all 2-bounded linear functional on X^2 and we call it the conjugate space of X^2.

Proposition (2.5):

$B(X^2,F)$ is a complete normed linear space.

Proof:

Let $\{T_k\}$ be a Cauchy sequence in $B(X^2,F)$ thus, $\lim_{k\to\infty}\|T_k - T_{k+p}\| = 0, \forall\, p = 1,2,\ldots$. Also,

$|(T_k - T_{k+p})(x_1,x_2)| \le \|T_k - T_{k+p}\|\|x_1,x_2\|$.Then

$|(T_k - T_{k+p})(x_1,x_2)| \longrightarrow 0$ as $k \longrightarrow \infty$, $\forall (x_1,x_2) \in X^2$.Thus $\{T_k(x_1,x_2)\}$ is a Cauchy sequence in F. Since $(F,\|.\|)$ is complete then $\lim_{k\to\infty} T_k(x_1,x_2) = y$ exists in $(F,\|.\|)$. Define $T: X^2 \to F$ by $T(x_1,x_2) = y$ then it can be easily verified that T is 2-bounded linear functional. Hence

$$|T_k(x_1,x_2) - T_{k+p}(x_1,x_2)| \le \|T_k - T_{k+p}\|\|x_1,x_2\| \le \varepsilon\|x_1,x_2\|$$

$\forall k \ge N(\varepsilon)$, $(x_1,x_2) \in X^2$, $p = 1,2,\ldots$. Letting $p \longrightarrow \infty$ we get $|T_k(x_1,x_2) - T(x_1,x_2)| \le \varepsilon\|x_1,x_2\|$, $\forall\, k \ge N(\varepsilon)$ and $\forall (x_1,x_2) \in X^2$. Thus

$$\sup\left\{\frac{|(T_k - T)(x_1,x_2)|}{\|x_1,x_2\|} : (x_1,x_2) \in X^2, \|x_1,x_2\| \ne 0\right\} \le \varepsilon.$$ Therefore

$\|T_k - T\| \le \varepsilon$, $\forall k \ge N(\varepsilon)$.Then $\|T_k - T\| \longrightarrow 0$ as $k \longrightarrow \infty$.

Hence $B(X^2, F)$ is complete.

3. Fuzzy Normed Linear Space and Fuzzy 2-Normed Linear Space:

In this section, some definitions and theorems are given which are used in this work.

Definition (3.1), [2]:

A fuzzy subset N of $X \times R$ is said to be a fuzzy norm on a linear space X in case for each $x, y \in X$ and $c \in F$, the following conditions hold:-

(FN_1) $N(x,t) = 0$ for each $t \le 0$;

(FN_2) $N(x,t) = 1$ for each $t > 0$ if and only if $x = 0$

(FN_3) If $0 \ne c \in F$ then $N(cx,t) = N(x, \frac{t}{|c|})$ for each $t > 0$.

(FN_4) $N(x+y, s+t) \ge \min\{N(x,s), N(y,t)\}$ for each $s, t \in R$.

(FN_5) $N(x,.)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x,t) = 1$

The pair (X,N) will be referred to as a fuzzy normed space

Definition (3.2), [2]:

Let (X,N) be a fuzzy normed space, a sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for each t>0. In this case, x is said to be limit of the sequence $\{x_n\}$ and we denote it by $\lim x_n$. Otherwise, the sequence is divergent.

Definition (3.3), [2]:

Let (X,N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be Cauchy sequence in case $\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1$ for each t>0 and p=1,2,….

Theorem (3.4), [2]:

Let (X,N) be a fuzzy normed space satisfying the following two conditions:

(FN_6) For each $t > 0$, $N(x,t) > 0$ implies $x = 0$.

(FN_7) For $x \neq 0$, $N(x,.)$ is a continuous function of R and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of R.

Let $\|x\|_\alpha = \inf\{t : N(x,t) \geq \alpha\}, \alpha \in (0,1)$ and $N' : X \times R \longrightarrow [0,1]$ be a function defined by

$$N'(x,t) = \begin{cases} \sup\{\alpha \in (0,1) : \|x\|_\alpha \leq t\} & \text{for } (x,t) \neq (0,0) \\ 0 & \text{for } (x,t) = (0,0) \end{cases}$$

Then, $N' = N$.

Definition (3.5), [6]:-

A fuzzy subset N of $X^2 \times R$ into [0,1] is said to be a fuzzy 2-norm on the linear space X in case the following axioms hold:

(FN_1) $N(x_1, x_2, t) = 0$ for each $t \leq 0$.

(FN_2) $N(x_1, x_2, t) = 1$ for each $t > 0$ if and only if x_1, x_2 are linearly dependent.

(FN_3) $N(x_1, x_2, t) = N(x_2, x_1, t)$.

(FN_4) If $0 \neq c \in R$ then $N(x_1, cx_2, t) = N(x_1, x_2, \frac{t}{|c|})$ for each $t > 0$.

(FN_5) for each $s, t \in R$

$N(x, z + y, s + t) \geq \min\{N(x,z,s), N(x,y,t)\}$

(FN_6) $N(x_1, x_2, .)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x_1, x_2, t) = 1$

The pair (X,N) will be referred to a fuzzy 2-normed space.

Theorem (3.6), [7]:

Let (X,N) be a fuzzy 2-normed space satisfying the following conditions:

(FN_7) For each $t > 0, N(x_1, x_2, t) > 0$ implies x_1, x_2 are linearly dependent.

(FN_8) For x_1, x_2 are linearly, $N(x_1, x_2, t)$ is a continuous function of $t \in R$ and strictly increasing in the subset $\{t : 0 < N(x_1, x_2, t) < 1\}$ of R.

Let $\|x_1, x_2\|_\alpha = \inf\{t : N(x_1, x_2, t) \geq \alpha\}$, $\alpha \in (0,1)$ and

$N' : X^2 \times R \longrightarrow [0,1]$ is defined by

$$N'(x_1, x_2, t) = \begin{cases} \sup\{\alpha \in (0,1) : \|x_1, x_2\|_\alpha \leq t\} & \text{when } x_1, x_2 \text{ are linearly independent}, t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

(a) $\{\|.,.\|_\alpha \mid \alpha \in (0,1)\}$ is an ascending family of $\alpha - 2 -$ norms corresponding to the fuzzy 2-normed space (X,N).

(b) (X, N') is a fuzzy $2-$normed space.

(c) $N' = N$

4.Fuzzy Conjugate Space of Fuzzy 2-Normed Space:

We start this section by the following remark.

Remark (4.1), [2]:

Let $(X,\|.\|)$ be a normed linear space. Define

$$N^*(x,t)=\begin{cases}0 & \text{if } t\le\|x\|,\ t\in R,\ x\in X\\ 1 & \text{if } t>\|x\|,\ t\in R,\ x\in X\end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

For each $x\in X$ and $t\in R$. Then (X,N^*) is a fuzzy normed linear space.

Definition (4.2):

Let $T:(X,N)\longrightarrow(F,N^*)$ be a linear functional where (X, N) be a fuzzy 2-normed linear space and N^* be a fuzzy norm defined in eq.(4.1). T is said to be fuzzy 2-bounded on X^2 in case there exists a positive number M such that,

$N^*(T(x_1,x_2),s)\ge N(x_1,x_2,\frac{s}{M})$, for each s>0. We denote the set of all fuzzy 2-bounded linear functional on X^2 by B^*.

Note (4.3):

Let T be a fuzzy 2-bounded linear functional on X^2. If x_1,x_2 are linearly dependent then $T(x_1,x_2)=0$.

Next, the definition of the uniformly bounded of linear operator with respect to the fuzzy normed space due to Bag and samanta appeared in [4]. With the aid of this definition we give the definition of uniformly 2-bounded of linear functional with respect to the fuzzy 2-normed space due to Narayauan and Vijayabalaji [3].

Definition (4.4):

Let $T:(X,N)\longrightarrow(F,N^*)$ be a linear functional where (X,N) be a fuzzy 2-normed linear space satisfying (FN_7) and (F,N^*) be a fuzzy normed space where N^* defined in eq.(4.1). T is said to be uniformly 2-bounded in case there exists M>0 such that $|T(x_1,x_2)| \leq M\|x_1,x_2\|_\alpha$ for each $\alpha \in (0,1)$.

Theorem(4.5):

Let (X,N) be a fuzzy 2-normed linear space satisfying (FN_7) and $T:(X,N) \longrightarrow (F,N^*)$ be a linear functional. If T is fuzzy 2-bounded linear functional then T is uniformly 2-bounded.

Proof:

Suppose that T is fuzzy 2-bounded linear functional thus there exists M>0 such that $N^*(T(x_1,x_2),s) \geq N(x_1,x_2,\frac{s}{M})$ for $s > 0$

Then $N^*(T(x_1,x_2),s) \geq N(x_1,Mx_2,s)$

Now $\|x_1,Mx_2\|_\alpha < t$ then $\inf\{s: N(x_1,Mx_2,s) \geq \alpha\} < t$

Thus there exists $s_0 < t$ such that $N(x_1,Mx_2,s_0) \geq \alpha$. This implies that $N^*(T(x_1,x_2),s_0) \geq \alpha$.

Hence $|T(x_1,x_2)| \leq s_0 < t$. Then

$|T(x_1,x_2)| \leq M\|x_1,x_2\|_\alpha$. This implies that T is uniformly 2-bounded .

Theorem (4.6):

B^* is a linear space.

Proof:

Let $T_1, T_2 \in B^*$, then there exist two positive numbers M_1 and M_2 such that for each $(x_1,x_2) \in X^2$.

$N^*(T_1(x_1,x_2),t) \geq N(x_1,x_2,\frac{t}{M_1}), t > 0$,

and

$$N^*(T_2(x_1,x_2,t) \geq N(x_1,x_2,\frac{t}{M_2}), t>0$$

Thus for any two non-zero scalars α and λ we have

$$N^*((\alpha T_1+\lambda T_2)(x_1,x_2),t)$$

$$\geq \min\{N^*(\alpha T_1(x_1,x_2),\frac{t}{2}), N^*(\lambda T_2(x_1,x_2),\frac{t}{2})\}$$

$$\geq \min\{N\left(x_1,x_2,\frac{t}{2|\alpha|M_1}\right), N\left(x_1,x_2,\frac{t}{2|\lambda|M_2}\right)\}$$

Choose $M = Max\{2|\alpha|M_1, 2|\lambda|M_2\}+1$. Thus

$M \geq 2|\alpha|M_1$ and $M \geq 2|\lambda|M_2$

This implies that $\frac{t}{2|\alpha|M_1} \geq \frac{t}{M}$ and $\frac{t}{2|\lambda|M_2} \geq \frac{t}{M}$ for each $t>0$. Hence

$$N\left(x_1,x_2,\frac{t}{2|\alpha|M_1}\right) \geq N\left(x_1,x_2,\frac{t}{M}\right)$$

and

$$N\left(x_1,x_2,\frac{t}{2|\lambda|M_2}\right) \geq N\left(x_1,x_2,\frac{t}{M}\right).$$

Therefore, $N^*((\alpha T_1+\lambda T_2)(x_1,x_2),t) \geq N\left(x_1,x_2,\frac{t}{M}\right)$ for each $t>0$.

This implies that $\alpha T_1+\lambda T_2 \in B^*$ for each α and $\lambda \in R$. Hence B^* is a linear space.

Note (4.7):

Let (X,N) be a fuzzy 2-normed linear space, we call B^*, the fuzzy conjugate space of X^2.

Theorem (4.8):

Let (X,N) be a fuzzy 2-normed linear space satisfying the conditions (FN_7) and (FN_8).

Then for any increasing (or decreasing) sequence $\{\alpha_k\}$ in (0,1), $\alpha_k \longrightarrow \alpha \in (0,1)$ implies $\|x_1, x_2\|_{\alpha_k} \longrightarrow \|x_1, x_2\|_\alpha$, for each $(x_1, x_2) \in X^2$

Proof:-

For x_1, x_2 are linearly dependent it is clear that $\alpha_k \longrightarrow \alpha$ implies $\|x_1, x_2\|_{\alpha_k} \longrightarrow \|x_1, x_2\|_\alpha$. Suppose x_1, x_2 are linearly independent and let $\{\alpha_k\}$ be an increasing sequence in (0,1) such that $\alpha_k \longrightarrow \alpha \in (0,1)$. Let $\|x_1, x_2\|_{\alpha_k} = t_k$ and $\|x_1, x_2\|_\alpha = t$. Then

$N(x_1, x_2, t_k) = \alpha_k$ and

$N(x_1, x_2, t) = \alpha$ ……………………………………………………(4.2)

Since $\{\|.,.\|_\alpha : \alpha \in (0,1)\}$ is an increasing family of 2-norms, $\{t_k\}$ is an increasing sequence of real numbers and it is bounded above by t. Hence $\{t_k\}$ is converge. Thus $\lim_{k\to\infty} N(x_1, x_2, t_k) = \lim_{k\to\infty} \alpha_k$ and

$N(x_1, x_2, \lim_{k\to\infty} t_k) = \alpha$ ……………………………………………………(4.3)

From eq.(4.2) and eq.(4.3) we have

$N(x_1, x_2, \lim_{k\to\infty} t_k) = N(x_1, x_2, t)$. Therefore, $\lim_{k\to\infty} \|x_1, x_2\|_{\alpha_k} = \|x_1, x_2\|_\alpha$

Similarly, if $\{\alpha_k\}$ is a decreasing sequence in (0,1) and $\{\alpha_k\} \longrightarrow \alpha \in (0,1)$, then it can be shown that $\|x_1, x_2\|_{\alpha_k} \longrightarrow \|x_1, x_2\|_\alpha$ for each $(x_1, x_2) \in X^2$.

Theorem (4.9):

Let X be a linear space and $\{\|.,.\|_\alpha : \alpha \in (0,1)\}$ be an ascending family of 2-norms on X. Assume further that for any increasing (or decreasing) sequence $\{\alpha_k\}$ in (0,1) and $\alpha_k \longrightarrow \alpha$ implies $\|x_1, x_2\|_{\alpha_k} \longrightarrow \|x_1, x_2\|_\alpha$ for each $(x_1, x_2) \in X^2$, define

$$N(x_1, x_2, t) = \begin{cases} \sup\{\alpha \in (0,1) : \|x_1, x_2\|_\alpha \leq t\} & \text{when } x_1, x_2 \text{ are linearly independent}, t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$
……………………………………………………………..…….(4.4)

and

$$\|x_1, x_2\|_\alpha' = \inf\{t : N(x_1, x_2, t) \geq \alpha\}, \alpha \in (0,1) \text{ ………..……………}(4.5)$$

Then $\|x_1, x_2\|_\alpha' = \|x_1, x_2\|_\alpha, \forall \alpha \in (0,1)$

Proof:

For x_1, x_2 linearly dependent it is clear that $\|x_1, x_2\|_\alpha' = \|x_1, x_2\|_\alpha = 0, \forall \alpha \in (0,1)$. Suppose x_1, x_2 are linearly independent. Let $\alpha_0 \in (0,1)$. Putting $\|x_1, x_2\|_{\alpha_0} = t_0$. Then $t_0 > 0$. From eq.(4.4) we get $N(x_1, x_2, t_0) \geq \alpha_0$. Now from eq.(4.5) we get

$$\|x_1, x_2\|_{\alpha_0}' \leq t_0 = \|x_1, x_2\|_{\alpha_0} \text{ ……..………………………………..…….}(4.6)$$

Next let $r > \|x_1, x_2\|_{\alpha_0}'$

Then there exists $t_1 < r$ such that $N(x_1, x_2, t_1) \geq \alpha_0$. Thus $\sup\{\alpha \in (0,1) : \|x_1, x_2\|_\alpha \leq t_1\} \geq \alpha_0$. If $\sup\{\alpha \in (0,1) : \|x_1, x_2\|_\alpha \leq t_1\} = \alpha_0$, then there exists an increasing sequence $\{\alpha_k\}$ in (0,1) such that $\alpha_k \uparrow \alpha_0$ and $\|x_1, x_2\|_{\alpha_k} \leq t_1$.

Thus $\|x_1, x_2\|_{\alpha_0} \leq t_1 < r$. On other hand, if $\sup\{\alpha \in (0,1) : \|x_1, x_2\|_\alpha \leq t_1\} > \alpha_0$

Then it follows that $\|x_1, x_2\|_{\alpha_0} \leq t_1 < r$.

Thus

$$\|x_1, x_2\|_{\alpha_0}' \geq \|x_1, x_2\|_{\alpha_0} \text{ ……………………………………...…..}(4.7)$$

From eq.(4.6) and eq.(4.7) we get $\|x_1, x_2\|'_{\alpha 0} = \|x_1, x_2\|_{\alpha 0}$ for each $(x_1, x_2) \in X^2$. Since $\alpha_0 \in (0,1)$ is arbitrary

$\|x_1, x_2\|'_{\alpha} = \|x_1, x_2\|_{\alpha} \ \forall (x_1, x_2) \in X^2, \alpha \in (0,1)$

Definition (4.10):-

Let (X,N) be a fuzzy 2-normed linear space satisfying (FN_7) and (FN_8) and T be a fuzzy 2-bounded linear functional, we define

$$\|T\|^*_\alpha = \sup\left\{ \frac{|T(x_1, x_2)|}{\|x_1, x_2\|_{1-\alpha}} : \forall\ \alpha \in (0,1) \text{ and } x_1, x_2 \text{ are linearly independent} \right\}$$

Again we define

$$N^{**}(T, s) = \begin{cases} \sup\{\alpha \in (0,1) : \|T\|^*_\alpha \le s\} & \text{for } (T, s) \ne (0,0) \\ 0 & \text{otherwise} \end{cases}$$

…………………………………………………………………......(4.8)

It is easy to check that N** is a fuzzy norm on B*

Lemma (4.11):-

Let (X,N) be a fuzzy 2-normed linear space satisfying (FN_7) and (FN_8), N* be the fuzzy norm on X defined in eq.(4.1) and T be a fuzzy 2-bounded linear functional. Let N** be the fuzzy norm on B* defined by eq.(4.8)

Let

$\|T\|^{**}_\alpha = \inf\{t : N^{**}(T, t) \ge \alpha\} \ \forall\ T \in B^*, \forall\ \alpha \in (0,1)$ ………………(4.9)

Then

$\|T\|^*_\alpha = \|T\|^{**}_\alpha$

Proof:

By theorem (4.8) we have for any increasing (or decreasing) sequence $\{\alpha_k\}$ in (0,1) such that $\alpha_k \to \alpha \in (0,1)$

$\|x_1, x_2\|_{1-\alpha_k} \to \|x_1, x_2\|_{1-\alpha}$, for each $(x_1, x_2) \in X^2$ …………..….(4.10)

If T=0 then $\|T\|_{\alpha}^{**} = 0 = \|T\|_{\alpha}^{*}$ for each $\alpha \in (0,1)$

If $T \neq 0$. Let $\alpha_0 \in (0,1)$ and putting

$\|T\|_{\alpha_0}^{*} = t_0$, then $N^{**}(T, t_0) \geq \alpha_0$

Hence from eq. (4.9) we have $\|T\|_{\alpha_0}^{**} < t_0 = \|T\|_{\alpha_0}^{*}$

Thus

$\|T\|_{\alpha_0}^{**} \leq \|T\|_{\alpha_0}^{*}$ ……………………………………………..……....(4.11)

Next, let r> $\|T\|_{\alpha_0}^{**}$

Thus there exists $t_1 < r$ such that $N^{**}(T, t_1) \geq \alpha_0$

Hence there exists $t_1 < r$ such that $\sup\{\alpha \in (0,1) : \|T\|^*_{\alpha} \leq t_1\} \geq \alpha_0$

If $\sup\{\alpha \in (0,1) : \|T\|_{\alpha}^{*} \leq t_1\} = \alpha_0$, then there exists a sequence $\{\alpha_k\}$ in (0,1) such that $\alpha_k \uparrow \alpha$ and $\|T\|_{\alpha_k}^{*} \leq t_1$ for each k. Therefore

$$\sup\left\{\frac{|T(x_1, x_2)|}{\|x_1, x_2\|_{1-\alpha_k}^{*}} : \forall \alpha_k \in (0,1) \text{ and } x_1, x_2 \text{ are linearly independent}\right\} \leq t_1, \forall k$$

Thus $\dfrac{|T(x_1, x_2)|}{\|x_1, x_2\|_{1-\alpha_k}} \leq t_1$ for each $(x_1, x_2) \in X^2$ and x_1, x_2 are linearly independent. Thus

$$\sup\left\{\frac{|T(x_1, x_2)|}{\|x_1, x_2\|_{1-\alpha_0}} : \forall \alpha \in (0,1) \text{ and } x_1, x_2 \text{ are linearly independent}\right\} \leq t_1, \forall k$$

Thus $\|T\|_{\alpha_0}^{*} \le t_1 < r$. Therefore $\|T\|_{\alpha_0}^{**} \ge \|T\|_{\alpha_0}^{*}$

If $\sup\left\{\alpha \in (0,1) : \|T\|_{\alpha_0}^{*} \le t_1\right\} > \alpha_0$, then this implies that, $\|T\|^{*}_{\alpha_0} \le t_1 < r$.
Thus

$$\|T\|_{\alpha_0}^{**} \ge \|T\|_{\alpha_0}^{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.12)$$

From (4.11)and (4.12) we have $\|T\|_{\alpha_0}^{*} = \|T\|_{\alpha_0}^{**}$ since $\alpha_0 \in (0,1)$ is arbitrary, thus $\|T\|_{\alpha}^{*} = \|T\|_{\alpha}^{**} \quad \forall T \in B^{*}, \forall \alpha \in (0,1)$.

<u>Theorem (4.12):</u>

Let (X,N) be a fuzzy 2-normed space satisfying (FN_7) and (FN_8). Then (B^{*}, N^{**}) is a complete fuzzy normed linear space.

<u>Proof:-</u>

Let $\{T_k\}$ be a Cauchy sequence in (B^{*}, N^{**}). Then $\lim_{k\to\infty} N^{**}(T_k - T_{k+p}, t) = 1$, for each $t > 0$, $p = 1,2,3,\ldots$

Thus

$$\|T_k - T_{k+p}\|_{\alpha}^{*} \to 0 \text{ as } k \to \infty \text{ and for each } \alpha \in (0,1) \ldots\ldots\ldots\ldots\ldots (4.13)$$

Now $|T_k(x) - T_{k+p}(x)| = |(T_k - T_{k+p})(x)| \le \|T_k - T_{k+p}\|_{\alpha}^{*} \|x_1, x_2\|_{1-\alpha}$

For each $\alpha \in (0,1)$ and $(x_1, x_2) \in X^2$

Then $|T_k(x) - T_{k+p}(x)| \to 0$ as $n \to \infty$ for each $(x_1, x_2) \in X^2$

Thus $\{T_k(x)\}$ is a Cauchy sequence in F. Since F is complete, so $\lim_{k\to\infty} T_k(x) = y$ exists in $(F, \|.\|)$. Since this is true for each (x_1, x_2) in X^2. Let $\lim_{k\to\infty} T_k(x_1, x_2) = T(x_1, x_2)$

It can be easily verified that T is fuzzy 2-bounded linear functional. Since $\|T_k - T_{k+p}\|_{\alpha}^{*} \to 0$ as $n \to \infty$ and $\forall \alpha \in (0,1)$. Thus for any $\varepsilon > 0$, $\exists N_{\alpha}(\varepsilon)$ such that $\|T_k - T_{k+p}\|_{\alpha}^{*} < \varepsilon, \forall\, k \ge N_{\alpha}(\varepsilon)$.

Then

$$\left|T_k(x_1,x_2)-T_{k+p}(x_1,x_2)\right|\le\left\|T_k-T_{k+p}\right\|_\alpha^*\left\|x_1,x_2\right\|_{1-\alpha}$$

$$<\in\left\|x_1,x_2\right\|_{1-\alpha},\ \forall\ k\ge N_\alpha(\varepsilon),\ \forall(x_1,x_2)\in X^2,\ P=1,2,3,....$$

letting $p\to\infty$ we get $\left|T_k(x_1,x_2)-T(x_1,x_2)\right|\le\varepsilon\left\|x_1,x_2\right\|_{1-\alpha}$

$\forall\ k\ge N_\alpha(\varepsilon)$ and $\forall(x_1,x_2)\in X^2$

Then

$\left\|T_k-T\right\|_\alpha^*\to 0$ as $k\to\alpha,\ \forall\alpha\in(0,1)$.

Therefore $\lim_{k\to\infty}N^{**}(T_k-T,t)=1,\ \forall\ t>0$

Hence (B^*,N^{**}) is complete.

References:-

[1] Gahler S., ***"Lineare 2-normierte Raume"***, Mathematische Nachrichter, Vol.4, PP.1-43, 1964.

[2] Bag T. and Samanta S., ***"Finite Dimensional Fuzzy Normed Linear Spaces"***, J. Fuzzy Math., Vol.11, PP. 687-705, 2003.

[3] Narayanan A. and Vijayabalaji S., ***"Fuzzy n-normed linear space"***, International Journal of Mathematics and Mathematical Sciences, Vol.24, PP.3963-3977, 2005.

[4] Bag T. and Samanta S., ***"Fuzzy bounded linear operator"***, Fuzzy sets and systems, Vol.151, PP. 513-547, 2005.

[5] Srivastava N., Bhattacharya S. and LAL S. ***"On Hahn-Banach extension of linear N-Functionals in N-Normed"***, Math .Maced. Vol.4, PP.25-32, 2006.

[6] Vijayabalaji S. and Thillaigovindan N, ***"Complete Fuzzy n-normed linear space"***, Journal of Fundamental Sciences, Vol.3, PP.119-126, 2007.

[7] Vijayabalaji S. and Thillaigovindan N.,***" Fuzzy n-inner product space"***, Bull. Kerean Math. Soc., Vol. 43, PP. 447-459, 2007.

حول الفضاءات المرافقة الضبابية للفضاءات 2- المعيارية الضبابية

م. د. فارعة علي جيجان
الجامعة المستنصرية – كلية العلوم – قسم الرياضيات

المستخلص:

في هذا البحث قدمنا تعريف الدالي الخطي 2-المقيد الضبابي كما تم تقد يم المعيار الضبابي له. وقدمنا ايضا الفضاء المرافق الضبابي للفضاء 2-المعياري الضبابي كما تم اعطاء بعض الحقائق المتعلقه به.

الكلمات المرشدة:- الدالي الخطي 2-المقيد الضبابي ، الفضاء المرافق الضبابي