On the Tensor Product for the Representations of the Dihedral Groups D_n

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حول الضرب الممتد لتمثيلات زمرة D_n

بحث مقدم من قبل

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Abstract

This paper covers our computational work and the algorithms designed for the determination of the tensor product of representations for the dihedral group D_n, which give us the general tensor product of representations for D_n, $\forall\, n \in Z_+$.

الخلاصة

في هذا العمل قمنا بحساب الضرب الممتد (Tensor) لتمثيلات الزمرة D_n عندما n عدد صحيح موجب وذلك من خلال تقديم بعض الخوارزميات المصممة لتحديد عملية الضرب الممتد (Tensor) لتمثيلات زمرة D_n.

1. Introduction

The tensor product of representations and the representations of dihedral group D_n has been given by [5], [7] respectively.

The importance of representation and character theory for the study of groups stems and should it be necessary to present a concrete description of a group, this can be achieved with a matrix representation.

In this work our focus will lie on the tensor product of representations for D_n, and we give algorithms designed of programs to calculate tensor product for D_n.

2. Preliminaries

In this section some definitions and basic concepts are introduced.

Definition (2-1) [5]

The **tensor product** for matrices $A = (a_{ij})$ and $B = (b_{ij})$ of degree n and m respectively defined by:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}_{nm \times mn}$$

Example

Consider $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \\ 4 & 2 & 3 \end{bmatrix}_{3\times 3}$ then:

$$A \otimes B = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 1 & 0 \\ 2 & 1 & -1 & -2 & -1 & 1 \\ 4 & 2 & 3 & -4 & -2 & -3 \\ \hline 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 4 & 2 & -2 \\ 0 & 0 & 0 & 8 & 4 & 6 \end{array}\right]_{6\times 6}$$

Proposition (2-2) [8]

Let $A, A^{\backslash} \in M_n\,(F)$ and $B, B^{\backslash} \in M_m\,(F)$ then:

1- $(A + A^{\backslash}) \otimes B = (A \otimes B) + (A^{\backslash} \otimes B)$

2- $(A \otimes B)\,(A^{\backslash} \otimes B^{\backslash}) = AA^{\backslash} \otimes BB^{\backslash}$

Definition (2-3) [1]

Let F be a field, the **general lineal group** GL(k, F) is a multiplicative group of all k × k matrices over the field F.

Definition (2-4) [3]

Let G be an arbitrary group, a **matrix representation** of G is homomorphism.

T: G → GL(k, F), such that

$T(xy) = T(x)\,T(y),\ \forall\, x, y \in G$

Where, the integer k is the degree (or dimension) of T, and
if T(e) = I, then I is the identity matrix of degree k.

Example

Let G be the cyclic group of degree 2, and write G = {1, x}

Then T: G → GL(2, R) where

$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a representation of G and the dimension of T is also 2.

Definition (2-5) [5]

Let T and S be two representations of degree k and ℓ respectively of the any group G then:

The tensor product of T and S defined by $K(g) = (T \otimes S)_{(g)}$

$= T(g) \otimes S(g), \forall\, g \in G$.

Hence

K(g) is representation of degree k × ℓ.

Definition (2-6) [7]

If T is a matrix representation of finite group G over a field F, **the character χ** is the mapping χ: G → F defined by

χ(g) = tr(T(g)), $, \forall\, g \in G$ where tr(T(g)) refers to the trace of the matrix T(g).

Example

In a cyclic group of degree 2, the representation T: C_2 → GL(2, R),

$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The character X of T is

$\chi(T_{(1)}) = 1 + 1 = 2$, and

$\chi(T_{(x)}) = 0 + 0 = 0$

Proposition (2-7) [5]

If A and B are n × n matrices then:

1- tr (AB) = tr (BA).

2- If A is non-singular then tr $(A^{-1}BA) = tr(B)$.

Proof

1- Let $A = (a_{ij})$ and $B = (b_{ij})$ with $1 \leq i, j \leq n$ then

$$tr(AB) = tr\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kj}$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{n} b_{kj}a_{ik} = tr\left(\sum_{i=1}^{n} b_{kj}a_{ik}\right)$$

$$= tr(BA)$$

2- By step (1) $tr(A^{-1}(BA)) = tr((BA)A^{-1})$

$$= tr(B(AA^{-1}))$$

$$= tr(B)$$

Definition (2-8) [6]

A **class function** on a group G is a function:

$f: G \rightarrow \mathbb{C}$ which is constant on conjugacy classes, that is:

$f(x^{-1}yx) = f(y), \ , \forall\ y \in G$.

Proposition (2-9) [5]

If χ is a character of G, then χ is a class function on G, that is if x and y are conjugate in G, then $\chi(x) = \chi(y)$.

Proof:

By hypothesis, there exists $g \in G$ such that $g^{-1} \times g = y$

Let T be the representation associated with χ, then

$\chi(y) = tr(T(y)) = tr\ T(g^{-1} \times g))$

$$= tr(T(g^{-1})(T(x)T(g)))$$

$$= tr(T(g^{-1})T(g)T(x))$$

$$= tr(T(g^{-1}g)T(x))$$

$$= tr(T(x)) = \chi(x)$$

Theorem (2-10) [6]

Sum and product of characters are character where χ and Ψ are characters of a group G, then

1- The sum of χ and Ψ will be defined by:

$(\chi + \Psi)(g) = \chi(g) + \Psi(g)$

2- The multiplication of χ and Ψ will be defined by:

$(\chi.\ \Psi)(g) = \chi(g).\ \Psi(g)$, for $g \in G$

3. The Algorithms

This part contains a collection of the computer-ready Fortran algorithms for many standard methods of number theory installed in our main program.

Algorithm 1	The number of degree of presentation for dihedral groups D_n
Algorithm 2	The tensor product of two representations for dihedral groups D_n
Algorithm 3	The tensor product of three representations for dihedral groups D_n
Algorithm 4	The character representations for dihedral groups D_n
Algorithm 5	The main algorithm of tensor product for dihedral groups D_n, $n \geq 4$.

Algorithm (1)

The number of degree of representation for dihedral groups D_n

Input: n (the degree of dihedral groups D_n)

Step 1: to evaluate k where

$T: D_n \rightarrow GL(n, R)$

Where n is the degree (dimension) of the T.

$$\text{If } A \in GL(n, R) \Rightarrow A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1k} \\ \vdots & & \vdots \\ \alpha_{k1} & \cdots & \alpha_{kk} \end{bmatrix}_{k \times k}$$

Step 2: Do p = 1 to k

Do q = 1 to k

PRINT PU(p, q)

END q-Loop

END p-Loop

Output: The number of degree of representation for dihedral groups D_n.

Example (1)

Consider the dihedral group D_4 of symmetries of the square in the Euclidean plane R^2, we may write:

$D_4 = <\alpha, \beta: \alpha^4 = \beta^2 = 1, \alpha\beta = \alpha^3\beta>$

The clockwise rotations of the square about the center and through angles of 90°, 180°, 270°, 360°, say, r_{90}, r_{180}, r_{270}, r_{360} respectively; reflections h and v about the horizontal and vertical axes, reflections d_1, d_2 about the diagonals.

Let us write $\alpha = r_{90}$ and $\beta = d_2$, then $\alpha^2 = r_{180}$, $\alpha^3 = r_{270}$, $\alpha^4 = r_{360}$, $\beta*\alpha = v$, $\beta*\alpha^2 = d_1$ and $\beta*\alpha^3 = h$. Also, note that $\beta*\alpha = \alpha^{-1}*\beta = \alpha^3*\beta$.

Thus we see that

$D_4 = \{e, \alpha, \alpha^2, \alpha^3, \beta, \beta*\alpha, \beta*\alpha^2, \beta*\alpha^3\}$

Let $T: D_4 \rightarrow GL(2, \mathbb{C})$ is a function, defined by

$$T(\alpha)=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \text{ and } T(\beta)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

We say

$$T(e)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$T(\alpha^2) = (T(\alpha))^2 = T(\alpha).T(\alpha)$

$$=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$T(\alpha^3) = (T(\alpha))^3 = (T(\alpha))^2T(\alpha)$

$$=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

Then,

$$T(\alpha^4)=(T(\alpha))^4=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=T(e)$$

$$T(\beta)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\Rightarrow T(\beta^2)=(T(\beta))^2=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=T(e)$$

And

$$T(\beta*\alpha)=T(\beta)T(\alpha)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$T(\alpha^3*\beta)=T(\alpha^3)T(\beta)=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

Then, $T(\beta*\alpha) = T(\alpha^3*\beta)$

Hence, we obtain

T is representations of degree 2.

Algorithm (2)

Two representations for dihedral groups D_n

Input: n (the degree of dihedral groups D_n)

Step 1: HL is the matrix of dimension kn×nk

L(0, 0) = 0

Do p = 1 to n

Do q = 1 to n

T(s) = A(p, q)

END q-Loop

END p-Loop

Step 2: Do p = 1 to k

Do q = 1 to k

Set T(S) = V(p, q)

END q-Loop
END p-Loop
Step 3: Call algorithm 1
Step 4: To evaluate L where
L(i, j) = U(p, q) * V
Step 5: Set L(1, 1) = U(1, 1) * V
L(1, 2) = U(1, 2) * V
⋮
L(p, n) = U(1, n) * V

$$\text{Where } V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{bmatrix}_{k\times k}$$

$$L = \begin{bmatrix} \ell_{11} & \ell_{12} & \cdots & \ell_{1nk} \\ \ell_{21} & \ell_{22} & \cdots & \ell_{2nk} \\ \vdots & \vdots & & \vdots \\ \ell_{nk1} & \ell_{nk2} & \cdots & \ell_{nkkn} \end{bmatrix}_{nk\times kn}$$

Output: The tensor product of representation of dihedral D_n C(kn, kn).

Example (2)

We take $T(\alpha) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $T(\alpha^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $T(\alpha^3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$T(\beta) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$T(\alpha) \otimes T(\beta) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]_{4\times 4}$$

$$T(\alpha^2) \otimes T(\alpha) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]_{4\times 4}$$

$$T(\alpha^3) \otimes T(\beta) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]_{4\times 4}$$

$$T(\alpha^2)\otimes T(\beta)=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=\left[\begin{array}{cc|cc}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]_{4\times4}$$

$$T(\beta)\otimes T(\alpha)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\left[\begin{array}{cc|cc}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{array}\right]_{4\times4}$$

$$T(\alpha)\otimes T(\alpha^2)=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\otimes\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=\left[\begin{array}{cc|cc}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right]_{4\times4}$$

$$T(\beta)\otimes T(\alpha^3)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\left[\begin{array}{cc|cc}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{array}\right]_{4\times4}$$

$$T(\beta)\otimes T(\alpha^2)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=\left[\begin{array}{cc|cc}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]_{4\times4}$$

$$T(\alpha^2)\otimes T(\alpha^3)=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\left[\begin{array}{cc|cc}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{array}\right]_{4\times4}$$

$$T(\alpha^3)\otimes T(\alpha^2)=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\otimes\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=\left[\begin{array}{cc|cc}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \hline -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{array}\right]_{4\times4}$$

<u>Algorithm (3)</u>

Three representations for dihedral groups D_n

Input: n (the degree of dihedral groups D_n)

Step 1: Call algorithm 2

Step 2: Do p = 1 to m

Do q = 1 to m
H(p, q)
END q-Loop
END p-Loop

Step 3: To evaluate R where
R(p, q) = L(p, q).H

Step 4: Set
R(1, 1) = L(1, 1)*H
R(1, 2) = L(1, 2)*H
Where

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1m} \\ h_{21} & h_{22} & \cdots & h_{2m} \\ \vdots & \vdots & & \vdots \\ h_{k1} & h_{k2} & \cdots & h_{mm} \end{bmatrix}_{m\times m}$$

$$E = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1f} \\ e_{21} & e_{22} & \cdots & e_{2f} \\ \vdots & \vdots & & \vdots \\ e_{f1} & e_{f2} & \cdots & e_{ff} \end{bmatrix}_{f\times f}$$

Where f = nk × m

Step 6: Do p = 1 to f
Do q = 1 to f
PRINT E(p, q)
END q-Loop
END p-Loop

Output: The three representations of dihedral group D_n E(f, f)

Example (3)

$$T(\alpha) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, T(\beta\alpha) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$T(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, T(\alpha^3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T(\alpha) \otimes T(\beta\alpha) \otimes T(\alpha) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]_{4\times 4} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{2\times 2}$$

$$
= \left[\begin{array}{cc|cc|cc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]_{8\times 8}
$$

$$
T(e)\otimes T(\beta\alpha)\otimes T(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
$$

$$
= \left[\begin{array}{cc|cc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right] \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
$$

$$
= \left[\begin{array}{cc|cc|cc|cc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{array}\right]_{8\times 8}
$$

$$
T(e)\otimes T(\alpha^3)\otimes T(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
$$

$$
= \left[\begin{array}{cc|cc}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right] \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}_{8\times 8}
$$

$$
T(e) \otimes T(\alpha) \otimes T(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}_{8\times 8}
$$

Algorithm (4)

The character of representation for dihedral groups D_n

Input: n (the degree of D_n)

Step 1: $\chi(0) = 0$

Step 2: Do p = 1 to k

$\chi(p)$

END p-Loop

Step 3: Do q = 1 to n

$\chi(q)$

END q-Loop

Step 4: Do p = 1 to k

Do q = 1 to n

$\chi(m) = \chi(p) * \chi(q)$

END q-Loop

END p-Loop
PRINT $\chi(m)$

Step 5: Set $\chi(m) = \begin{bmatrix} \chi_{(1)} \\ \chi_{(2)} \\ \chi_{(3)} \\ \vdots \\ \chi_{(f)} \end{bmatrix}$

f = (n.k)/2

Step 6: Call algorithm 3

Step 7: Call algorithm 4

Output: The character of representation of dihedral groups D_n

$\chi(m)$, m = 1 to f

Example (4)

The character table of representation for D_4 is

g	1	α^2	α	β	$\alpha\beta$
$\lvert C_{D_4}(g) \rvert$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

In 1

$\chi(1)\otimes \chi(2) = 1 * 1 = 1$
$\chi(1)\otimes \chi(3) = 1 * 1 = 1$
$\chi(1)\otimes \chi(4) = 1 * 1 = 1$
$\chi(1)\otimes \chi(5) = 1 * 2 = 2$
$\chi(2)\otimes \chi(3) = 1 * 1 = 1$
$\chi(2)\otimes \chi(4) = 1 * 1 = 1$
$\chi(2)\otimes \chi(5) = 1 * 2 = 2$
$\chi(3)\otimes \chi(4) = 1 * 1 = 1$
$\chi(3)\otimes \chi(5) = 1 * 2 = 2$
$\chi(4)\otimes \chi(5) = 1 * 2 = 2$

In α^2

$\chi(1)\otimes \chi(2) = 1 * 1 = 1$
$\chi(1)\otimes \chi(3) = 1 * 1 = 1$
$\chi(1)\otimes \chi(4) = 1 * 1 = 1$
$\chi(1)\otimes \chi(5) = 1 * (-2) = -2$
$\chi(2)\otimes \chi(3) = 1 * 1 = 1$

$\chi(2) \otimes \chi(4) = 1 * 1 = 1$
$\chi(2) \otimes \chi(5) = 1 * (-2) = -2$
$\chi(3) \otimes \chi(4) = 1 * 1 = 1$
$\chi(3) \otimes \chi(5) = 1 * (-2) = -2$
$\chi(4) \otimes \chi(5) = 1 * (-2) = -2$

In α
$\chi(1) \otimes \chi(2) = 1 * 1 = 1$
$\chi(1) \otimes \chi(3) = 1 * -1 = -1$
$\chi(1) \otimes \chi(4) = 1 * -1 = -1$
$\chi(1) \otimes \chi(5) = 1 * 0 = 0$
$\chi(2) \otimes \chi(3) = 1 * -1 = -1$
$\chi(2) \otimes \chi(4) = 1 * -1 = -1$
$\chi(2) \otimes \chi(5) = 1 * 0 = 0$
$\chi(3) \otimes \chi(4) = -1 * -1 = 1$
$\chi(3) \otimes \chi(5) = -1 * 0 = 0$
$\chi(4) \otimes \chi(5) = -1 * 0 = 0$

In β
$\chi(1) \otimes \chi(2) = 1 * -1 = -1$
$\chi(1) \otimes \chi(3) = 1 * 1 = 1$
$\chi(1) \otimes \chi(4) = 1 * -1 = -1$
$\chi(1) \otimes \chi(5) = 1 * 0 = 0$
$\chi(2) \otimes \chi(3) = -1 * 1 = -1$
$\chi(2) \otimes \chi(4) = -1 * -1 = 1$
$\chi(2) \otimes \chi(5) = -1 * 0 = 0$
$\chi(3) \otimes \chi(4) = 1 * -1 = -1$
$\chi(3) \otimes \chi(5) = 1 * 0 = 0$
$\chi(4) \otimes \chi(5) = -1 * 0 = 0$

In $\alpha\beta$
$\chi(1) \otimes \chi(2) = 1 * -1 = -1$
$\chi(1) \otimes \chi(3) = 1 * -1 = -1$
$\chi(1) \otimes \chi(4) = 1 * 1 = 1$
$\chi(1) \otimes \chi(5) = 1 * 0 = 0$
$\chi(2) \otimes \chi(3) = -1 * -1 = 1$
$\chi(2) \otimes \chi(4) = -1 * 1 = -1$
$\chi(2) \otimes \chi(5) = -1 * 0 = 0$
$\chi(3) \otimes \chi(4) = -1 * 1 = -1$
$\chi(3) \otimes \chi(5) = -1 * 0 = 0$
$\chi(4) \otimes \chi(5) = 1 * 0 = 0$

Algorithm (5)

The algorithm of the main program

The tensor product of representation for dihedral groups D_n

$1 \leq n \leq 6$

Input: n (the degree of the dihedral groups)

Step 1: Call algorithm 1

Step 2: Call algorithm 2

Step 3: Call algorithm 3

Step 4: Call algorithm 4

Output: (T(p), p = 1 to k) to evaluate the tensor product of representation for dihedral groups D_n

End

Proposition

The $\bigotimes_{i=1}^{n} T$ it is true, $\forall\, n \in Z_+$.

Proof

We prove by an inductive argument the statement is certainly true for K = 1 (call Algorithm (1)).

Assuming it holds for an arbitrary K, then $\bigotimes_{i=1}^{K} T$ (call Algorithm (5)).

We must show that it also holds for K + 1. but this is immediate fron Algorithm (1) and Algorithm (5) imply $\bigotimes_{i=1}^{K+1} T$.

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Recommendations

The Recommendations for future work are:

1- Evaluation the tensor product for representation for groups
$C_n. = < x: \ x^{n} = e >$

2- Evaluating the tensor product of representation of the alternating group A_n.