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Tsallis Entropy Perspective on Multivariate Controlled Autoregressive Fractional Integrated Moving Average System

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Abstract

In this study, we have investigated Tsallis entropy of a multivariate controlled autoregressive fractional integrated moving average (MCARFIMA) system. A characteristic function of this system state is derived using the corresponding its residual characteristic functions.

Furthermore, an analytical expression for the Tsallis entropy of system process is established under independent condition between the control process and moving average. An algorithm is proposed to calculate the values of Tsallis entropy of system process when it is influenced by Gaussian and Cauchy processes. The results are discussed and analyzed to demonstrate the fractional integrated on the system's informational structure and dynamical behaviour. Finally, an example is given to illustrative the proposed algorithm and information behaviour of proposed system under Gaussian process.

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1. Introduction

The multivariate autoregressive moving average (MARMA) models and their generalizations are widely used for analyzing the structure of time series and for forecasting future values. One of these generalizations is MCARFIMA system, which is characterized by the process of fractional integrated and control process. This model has been used throughout various fields, such as energy Nichiforov (2017) [1], economics Ugoh et.al. (2021)[2], industry Wang et.al. (2021) [3], trade and sales Wang et.al. (2013) [4], medicine and health Chadsuthi et.al. (2015) [5], insurance and social security Ulyah and Mardianto (2019)[6].

As brief historical review let us begin with the early approaches, Jenkins and Priestley (1957)[7] obtained the characteristic function of autoregressive (AR(2)) state in the univariate case from its residual functions, while Abid (2006) [8] generalized this study to represent the characteristic function of ARMA(p,q) process. Davis et.al. (2021) [9] estimated the parameters of ARIMA model by minimizing the integrated weighted mean squared error between the empirical and simulated characteristic function, when the true characteristic functions cannot be explicitly computed. Sabzikar et.al. (2019) [10] introduced the autoregressive tempered fractionally integrated moving average ARTFIMA model, which incorporates tempering to address the non-summability of ARFIMA's covariance function, making it more mathematically tractable. Gamage [11] et.al. (2019) proposed technique of estimation for seasonal ARFIMA states by using the empirical characteristic function under stable innovations.

On the other hand, entropy, as a fundamental measure of information, plays a crucial role in characterizing the structure and nature of relationships between time series. It serves as a quantitative tool for assessing the extent to which knowledge of one variable reduces the uncertainty about another, thereby providing insights into the underlying dependencies, dynamics, and potential predictability within complex systems. Dubnov and Boulytchev (2023) [12] introduced an accelerated maximum entropy method for estimating the probabilistic parameters of time series models from observed data. This approach provides a computationally efficient framework for entropy estimation in ARMA models. Similarly, McElroy (2024) [13] applied maximum entropy optimization with sparse regularization to estimate ARMA parameters and characterize the system's statistical properties. Zorzi (2024) [14] extended maximum entropy techniques to graphical ARMA models, highlighting their effectiveness in probabilistic modeling and parameter estimation . Abid et.al. (2025) [15] introduced the characteristic function of MCARMA model from the terms of its residual characteristic function. Also, they determined the explicit formula to compute the Rényi entropy of output state for Gaussian, Cauchy and Laplace processes. In the present work, we generalize this study by incorporating a fractional integration into the system, which increases its behavioral complexity and alters its informational structure. Consequently, a more flexible type of entropy is required; Tsallis entropy was selected for this purpose, as its effectiveness has been demonstrated through the obtained results.

The remainder of this paper is organized as follows. Section 2 presents the basic concepts and Preliminaries. Section 3 presents the system description. In Section 4, we derived a characteristic function of MCARFIMA state. In Section 5, we introduce the formula and algorithm to compute the Tsallis entropy, while Section 6 concludes the paper by summarizing the key findings and insights.

2. Preliminaries

Consider $\mathrm{I_d}$ is the identity matrix of size d, $\mathrm{MG_d}$ and $\mathrm{MC_d}$ are the distributions of multivariate Gaussian, and multivariate Cauchy, respectively, $\mathbb{R}^{\mathrm{d}}$ is a d-dimensional space, and the notation $(')$ represents matrix transpose. Assume that $P(y;\vartheta)$ represents the probability distributions of random variable y parameterized by ϑ. The Shannon entropy of y , is defined as follows Shannon (1948) [16]:

$$^{S}H(y) = -E\left(ln\big(P(y;\vartheta)\big)\right) \tag{1}$$

The Tsallis entropy is a generalization of Shannon entropy, introduced to handle non extensive systems where long-range interaction or correlations exist. For a random variable y , it is defined as follows Anastasiadis (2012) [17]:

$$^{T}H_{\beta}(y) = \begin{cases} \dfrac{1}{\beta-1}\Big(1 - E\big(P(y;\vartheta)^{\beta-1}\big)\Big), 0 < \beta \neq 1, \\ \quad {}^{S}H(y) \qquad\qquad\qquad\qquad , \ \beta = 1 \end{cases} \tag{2}$$

where, β is called the entropic index or Tsallis parameter. The relationship between Tsallis entropy and Shannon entropy is given by the following expression Behera et.al. (2024)[18] $\lim_{\beta\to 1} {}^{T}H_{\beta}(y) = {}^{S}H(y)$.

The multivariate Gaussian and Cauchy distributions are used to describe random data in complicated systems. The Gaussian distribution is characterized by its regularity and dependence on the mean and covariance, while the Cauchy distribution has heavy tails, allowing it to model outliers and high perturbations.

Lemma 1 Cover (1999)[19] If $y \in \mathbb{R}^{\mathrm{d}}$ is a random vector with expected value $E(y) = 0$ and covariance matrix $\mathbf{\Sigma}_{\mathrm{y}}$ then, regardless of the distribution of y (it does not need to be Gaussian distributed), then the following inequality holds

$$^{S}H(y) \leq \frac{1}{2}\ln\left((2\pi e)^{d}\det\left(\mathbf{\Sigma}_{\mathrm{y}}\right)\right) \tag{3}$$

with equality if and only if $\mathrm{y} \sim \mathrm{MG}_{\mathrm{d}}(0, \mathbf{\Sigma}_{\mathrm{y}})$.

In the context of this work, and in order to determine a more general formulation of boundedness of entropy measure, we first rely on the established relationship between Rènyi entropy and Tsallis Mariz (1992)[20], and then incorporate the boundedness property of Rényi entropy (lemma 2. [15]). Based on this, we obtain the following lemma.

Lemma 2 Abid (2025) [15] Suppose that a random vector $y \in \mathbb{R}^{\mathrm{d}}$ has covariance matrix $\mathbf{\Sigma}_{\mathrm{y}}$, then the bound of Tsallis entropy can be expressed in terms of $\mathbf{\Sigma}_{\mathrm{y}}$ as:

$$^{T}H_{\beta}(Z) \leq \frac{1}{(\beta-1)}\left(1 - exp\left((1-\beta)\left(\mathcal{C}_{d}(\beta) + \frac{1}{2}\ln\left(\det\left(\mathbf{\Sigma}_{\mathrm{y}}\right)\right)\right)\right)\right) \tag{4}$$

where,

$$\mathcal{C}_{\mathrm{d}}(\beta) = \begin{cases} \frac{\mathrm{d}}{2}\ln\left(\frac{\pi(\beta(\mathrm{d}+2)-\mathrm{d})}{\beta-1}\right) + \frac{1}{\beta-1}\ln\left(\frac{(\beta(\mathrm{d}+2)-\mathrm{d})}{2\beta}\right) + \ln\left(\frac{\Gamma\left(\frac{\beta}{\beta-1}\right)}{\Gamma\left(\frac{(\beta(\mathrm{d}+2)-\mathrm{d})}{2(\beta-1)}\right)}\right), & \beta > 1 \\ \frac{\mathrm{d}}{2}\ln\left(\frac{\pi(\beta(\mathrm{d}+2)-\mathrm{d})}{1-\beta}\right) - \frac{\beta}{1-\beta}\ln\left(\frac{(\beta(\mathrm{d}+2)-\mathrm{d})}{2\beta}\right) - \ln\left(\frac{\Gamma\left(\frac{\beta}{1-\beta}\right)}{\Gamma\left(\frac{(\beta(\mathrm{d}+2)-\mathrm{d})}{2(1-\beta)}\right)}\right), & \frac{\mathrm{d}}{\mathrm{d}+2} < \beta < 1 \\ \frac{\mathrm{d}}{2}\ln(2\pi\mathrm{e}), & \beta = 1 \end{cases}$$

$$\mathcal{C}_{\mathrm{d}}(\beta) = \begin{cases} \frac{\mathrm{d}}{2}\ln\left(\frac{\pi(\beta(\mathrm{d}+2)-\mathrm{d})}{\beta-1}\right) + \frac{1}{\beta-1}\ln\left(\frac{(\beta(\mathrm{d}+2)-\mathrm{d})}{2\beta}\right) + \ln\left(\frac{\Gamma\left(\frac{\beta}{\beta-1}\right)}{\Gamma\left(\frac{(\beta(\mathrm{d}+2)-\mathrm{d})}{2(\beta-1)}\right)}\right), & \beta > 1 \\ \frac{\mathrm{d}}{2}\ln\left(\frac{\pi(\beta(\mathrm{d}+2)-\mathrm{d})}{1-\beta}\right) - \frac{\beta}{1-\beta}\ln\left(\frac{(\beta(\mathrm{d}+2)-\mathrm{d})}{2\beta}\right) - \ln\left(\frac{\Gamma\left(\frac{\beta}{1-\beta}\right)}{\Gamma\left(\frac{(\beta(\mathrm{d}+2)-\mathrm{d})}{2(1-\beta)}\right)}\right), & \frac{\mathrm{d}}{\mathrm{d}+2} < \beta < 1 \\ \frac{\mathrm{d}}{2}\ln(2\pi\mathrm{e}), & \beta = 1 \end{cases}$$

And Γ is the gamma function.

Proposition 3 Let $P(y; \theta, \mathbf{\Sigma}_{\mathrm{y}})$ be a location-scale probabilistic model, where θ is the location parameter and $\mathbf{\Sigma}_{\mathrm{y}}$ is the scale parameter. In this case, the Tsallis entropy does not depend on the location parameter θ , but depends solely on the scale parameter $\mathbf{\Sigma}_{\mathrm{y}}$.

Proof: Begin by expressing the probability density function of the location-scale model

$$P(y; \theta, \mathbf{\Sigma}_{\mathrm{y}}) = \left(\det\left(\mathbf{\Sigma}_{\mathrm{y}}\right)\right)^{-\frac{1}{2}} P\left(\mathbf{\Sigma}_{\mathrm{y}}^{-\frac{1}{2}}(\mathrm{y}-\theta), 0, \mathrm{I}_{d}\right)$$

Consequently,

$$ {}^{T}H_{\beta}(y)=\frac{1}{\beta-1}\left(1-\left(\det(\boldsymbol{\Sigma}_{\mathrm{y}})\right)^{-\frac{\beta-1}{2}}E\left(P\left(\boldsymbol{\Sigma}_{\mathrm{y}}{}^{-\frac{1}{2}}(\mathrm{y}-\theta),0,\mathrm{I}_{d}\right)\right)\right),\beta\neq 1 $$

Hence, the Tsallis entropy is independent of the parameter θ.

Directly, from the definition of the Tsallis entropy, we obtain the following lemmas

Lemma 4 Consider $\mathrm{y}\sim \mathrm{M}G_{\mathrm{d}}(\theta,\boldsymbol{\Sigma}_{\mathrm{y}})$. Then,

$$ {}^{T}H_{\upbeta}(y)=\frac{\upbeta^{\frac{d}{2}}}{(1-\upbeta)}\left(\det(2\pi\boldsymbol{\Sigma}_{\mathrm{y}})\right)^{\frac{1-\upbeta}{2}}-\frac{1}{(1-\upbeta)} \tag{1} $$

Lemma 5 Consider $\mathrm{y}\sim \mathrm{MC}_{\mathrm{d}}(\theta,\boldsymbol{\Sigma}_{\mathrm{y}})$. Then,

$$ {}^{T}H_{\upbeta}(y)=\frac{1}{(1-\upbeta)}\left(\frac{\upbeta^{-\frac{\mathrm{d}}{1-\upbeta}}\Gamma\left(\frac{\mathrm{d}+1}{2}\right)\det(4\pi\boldsymbol{\Sigma}_{\mathrm{y}})^{\frac{1}{2}}}{\sqrt{\pi}}\right)^{(1-\upbeta)}-\frac{1}{(1-\upbeta)} \tag{2} $$

3. System Description

The multivariate controlled ARMA system (MCARMA) has long been considered a fundamental model class for multivariate time series. Bao et al.(2011) [21] introduced a stochastic control system; it is the multivariate controlled ARMA system. We propose a novel extension of this system by taking the fractional orders. This development will enhance the system's ability to model complex dynamics and improve its performance. The proposed system would take the following form:

$$ \left.\begin{array}{c} \mathrm{C}(\mathcal{L})(\mathrm{I}_{\mathrm{d}}-\mathcal{L})^{\gamma}\mathrm{y}(\mathrm{t})=\mathrm{D}(\mathcal{L})u(\mathrm{t})+\mathrm{E}(\mathcal{L})\varepsilon(\mathrm{t}) \\ \mathrm{y}(0)=0 \end{array}\right\} \tag{3} $$

where,

$$ \mathrm{C}(\mathcal{L})=\mathrm{I}_{\mathrm{d}}+\mathrm{C}_{1}\mathcal{L}+\mathrm{C}_{2}\mathcal{L}^{2}+\cdots+\mathrm{C}_{\mathrm{p}}\mathcal{L}^{\mathrm{p}} \tag{4} $$

$$ \mathrm{D}(\mathcal{L})=\mathrm{I}_{\mathrm{d}}+\mathrm{D}_{1}\mathcal{L}+\mathrm{D}_{2}\mathcal{L}^{2}+\cdots+\mathrm{D}_{\mathrm{r}}\mathcal{L}^{\mathrm{r}} \tag{5} $$

$$ \mathrm{E}(\mathcal{L})=\mathrm{I}_{\mathrm{d}}+\mathrm{E}_{1}\mathcal{L}+\mathrm{E}_{2}\mathcal{L}^{2}+\cdots+\mathrm{E}_{\mathrm{q}}\mathcal{L}^{\mathrm{q}} \tag{6} $$

where, $p,r,q\in\mathbb{N}$, $\mathrm{y}(\mathrm{t})$ is the system output state, $u(\mathrm{t})$ is the system input vector, $\varepsilon(\mathrm{t})$ is the white noise vector with mean zero, $\gamma\geq 0$ is fractional parameter. $\mathrm{C}_{\mathrm{i}}\in\mathbb{R}^{\mathrm{d}\times\mathrm{d}}$, $\mathrm{D}_{\mathrm{j}}\in\mathbb{R}^{\mathrm{d}\times\mathrm{d}}$, $\mathrm{E}_{\mathrm{k}}\in\mathbb{R}^{\mathrm{d}\times\mathrm{d}}$. Assume that $\mathrm{E}\left(u(\mathrm{t})u'(\mathrm{s})\right)=\begin{cases}\boldsymbol{\Sigma}_{u}, & \mathrm{s}=\mathrm{t}\\ 0, & \mathrm{s}\neq\mathrm{t}\end{cases}$,

$\mathrm{E}\left(\varepsilon(\mathrm{t})\varepsilon'(\mathrm{s})\right)=\begin{cases}\boldsymbol{\Sigma}_{\varepsilon}, & \mathrm{s}=\mathrm{t}\\ 0, & \mathrm{s}\neq\mathrm{t}\end{cases}$

and the both processes $u(\mathrm{t})$, $\varepsilon(\mathrm{s})$ are independent for any s, t.

Each term $(\mathrm{I}_{\mathrm{d}}-\mathcal{L})^{\gamma}$ is defined by the following Wold expansion (or MA representation):

$$ (\mathrm{I}_{\mathrm{d}}-\mathcal{L})^{\gamma}=\sum\nolimits_{m=0}^{\infty}\Theta_{m}\mathcal{L}^{m} \tag{11} $$

with coefficients Chung (2001) [22]

$$ \Theta_{m}=\frac{\Gamma(\gamma+1)}{\Gamma(m)\Gamma(\gamma-m+1)},\quad m=1,2,\ldots d, \tag{7} $$

Assume that $\mathrm{C}_{0}=\mathrm{D}_{0}=\mathrm{E}_{0}=\mathrm{I}_{\mathrm{d}}$, therefore, equation (7) becomes

$$ \left(\sum_{\mathrm{i}=0}^{\mathrm{p}}\mathrm{C}_{\mathrm{i}}\mathcal{L}^{\mathrm{i}}\sum\nolimits_{m=0}^{\infty}\Theta_{m}\mathcal{L}^{m}\right)\mathrm{y}(\mathrm{t})=\mathrm{D}(\mathcal{L})u(\mathrm{t})+\mathrm{E}(\mathcal{L})\varepsilon(\mathrm{t}) \tag{8} $$

It is important to refer thar when the value of fractional order $\gamma = 0$, the system (7) is reduced to MCARMA system [21][23] and, if $\mathbf{D}_i = \mathbf{0}$, the system (7) is reduced to MARFIMA model, which is stationary and invertible if $|\gamma| < 1/2$, (γ can be fractional number). On the left- hand side of the previous, since the series extends to infinity, it is truncated at N, where, it represents the number of terms taken for numerical approximation, hence

$$\left(\sum_{\mathrm{i}=0}^{\mathrm{p}} \mathrm{C_i}\mathcal{L}^{\mathrm{i}} \sum_{m=0}^{\mathrm{N}} \Theta_m \mathcal{L}^m\right)\mathrm{y(t)} = \mathrm{D}(\mathcal{L})u(\mathrm{t}) + \mathrm{E}(\mathcal{L})\varepsilon(\mathrm{t}) \tag{9}$$

The left hand side of above equation can be written as

$$\left(\sum_{\mathrm{i}=0}^{\mathrm{p}} \mathrm{C_i}\mathcal{L}^{\mathrm{i}} \sum_{m=0}^{\mathrm{N}} \Theta_m \mathcal{L}^m\right)\mathrm{y(t)} = \sum_{m=0}^{\mathrm{p+N}} \sum_{\substack{i=0\\ i\le p,\\ m-i\le \mathrm{N}}}^{m-1} \mathrm{C_i}\,\Theta_{m-i}\mathcal{L}^m \mathrm{y(t)}$$

As a simplified form, equation (14) can be written as

$$\left(\mathrm{I}_d - \sum_{m=1}^{\mathrm{p+N}} \Omega_m^C\,\mathcal{L}^m\right)\mathrm{y(t)} = \mathrm{D}(\mathcal{L})u(\mathrm{t}) + \mathrm{E}(\mathcal{L})\varepsilon(\mathrm{t}) \tag{10}$$

where,

$$\Omega_m^C = \sum_{\substack{i=0\\ i\le p,\\ m-i\le \mathrm{N}}}^{m} \mathrm{C_i}\,\Theta_{m-i} \tag{11}$$

Define the operator $\Omega^C\,(\mathcal{L})$ as

$$\Omega^C\,(\mathcal{L}) = \sum_{m=1}^{\mathrm{p+N}} \Omega_m^C\,\mathcal{L}^m \tag{12}$$

Consequently, the equation (7) becomes

$$\left(\mathrm{I}_d - \Omega^C\,(\mathcal{L})\right)\mathrm{y(t)} = \mathrm{D}(\mathcal{L})u(\mathrm{t}) + \mathrm{E}(\mathcal{L})\varepsilon(\mathrm{t}) \tag{13}$$

The conditions of stationarity and invertibility of $\mathrm{D}(\mathcal{L})$ and $\mathrm{E}(\mathcal{L})$coefficients are given. Then, the system (7) can be written using the Wold representation as

$$\mathbf{y}(t) = \mathcal{M}(\mathcal{L})u(t) + \widetilde{\mathcal{M}}(\mathcal{L})\varepsilon(t), \tag{14}$$

where,

$$\mathcal{M}(\mathcal{L}) = \left(\mathrm{I}_d - \Omega^C\,(\mathcal{L})\right)^{-1}\mathrm{D}(\mathcal{L}) = \sum_{\mathrm{i}=0}^{\infty} \mathcal{M}_{\mathrm{i}}\,\mathcal{L}^{\mathrm{i}} \tag{15}$$

$$\widetilde{\mathcal{M}}(\mathcal{L}) = \left(\mathrm{I}_d - \Omega^C\,(\mathcal{L})\right)^{-1}\mathrm{E}(\mathcal{L}) = \sum_{\mathrm{j}=0}^{\infty} \widetilde{\mathcal{M}}_j\,\mathcal{L}^{\mathrm{j}} \tag{16}$$

with $\mathcal{M}_0 = \widetilde{\mathcal{M}}_0 = \mathrm{I}_d$. Note in (2) that $\widetilde{\mathcal{M}}(\mathcal{L})$ is the polynomial with coefficients given by the impulse response function of a MARFIMA model [22]; however, the term $\mathcal{M}(\mathcal{L})u(t)$ depends on time t (not constant) and thus $\mathbf{y}(t)$ is not wide-sense stationary.

Now, we compute the matrices $\mathcal{M}_i, i = 0,1,2,\dots$, $\widetilde{\mathcal{M}}_j, j = 0,1,2,\dots$, from equation (19), we have

$$\left(\mathrm{I} - \Omega^{C}\left(\mathcal{L}\right)\right) \mathrm{y(t)} = \left(\mathrm{I} - \Omega^{C}\left(\mathcal{L}\right)\right) \sum_{\mathrm{i=0}}^{\infty} \mathcal{M}_{\mathrm{i}}\, \mathcal{L}^{\mathrm{i}}\, u(\mathrm{t}) + \left(\mathrm{I} - \Omega^{C}\left(\mathcal{L}\right)\right) \sum_{\mathrm{j=0}}^{\infty} \widetilde{\mathcal{M}}_{j}\ \varepsilon(\mathrm{t}) \tag{17}$$

Consequently, the last equation can be written as

$$\begin{aligned} &\left(\mathrm{I} - \Omega^{C}\left(\mathcal{L}\right)\right) \mathrm{y(t)} \\ &= \left(\mathrm{I_d} + \sum_{\mathrm{i=1}}^{\infty} \left(\mathcal{M}_{\mathrm{i}} - \sum_{\mathrm{s=1}}^{\mathrm{i}} \Omega_{s}^{C}\, \mathcal{M}_{\mathrm{i-s}} \right) \mathcal{L}^{\mathrm{i}} \right) + \quad \left(\mathrm{I_d} + \sum_{\mathrm{j=1}}^{\infty} \left(\widetilde{\mathcal{M}}_{j} - \sum_{\mathrm{s=1}}^{\mathrm{j}} \Omega_{s}^{C}\, \widetilde{\mathcal{M}}_{\mathrm{j-s}} \right) \mathcal{L}^{\mathrm{j}} \right) \varepsilon(\mathrm{t}) \end{aligned} \tag{18}$$

Again, if $\Omega_{s}^{C} = 0$ for $\mathrm{s} > \mathrm{p} + \mathrm{N}$, $\mathrm{D}_{s} = 0$ for $\mathrm{s} > \mathrm{r}$, $\mathrm{E_s} = 0$ for $\mathrm{s} > \mathrm{q}$, then equation (18) becomes

$$\left(\mathrm{I} - \Omega^{C}\left(\mathcal{L}\right)\right) \mathrm{y(t)} = \sum\nolimits_{\mathrm{i=0}}^{\infty} \mathrm{D}_{i}\, \mathcal{L}^{i} u(\mathrm{t}) + \sum\nolimits_{\mathrm{j=0}}^{\infty} \mathrm{E}_{j}\, \mathcal{L}^{j} \varepsilon(\mathrm{t}) \tag{19}$$

By comparing the coefficients of equations (23) , (24), we get

$$\mathcal{M}_{\mathrm{i}} = \mathrm{D}_{i} + \sum_{\mathrm{s=1}}^{\mathrm{i}} \Omega_{s}^{C}\, \mathcal{M}_{\mathrm{i-s}} \quad , i = 1,2, \dots , r \tag{20}$$

$$\widetilde{\mathcal{M}}_{j} = \mathrm{E}_{j} + \sum_{\mathrm{s=1}}^{\mathrm{j}} \Omega_{s}^{C}\, \widetilde{\mathcal{M}}_{\mathrm{j-s}} , j = 1,2, \dots , q \tag{21}$$

3. Characteristic Function

The characteristic function of MCARFIMA process can be represented based on its residual characteristic functions and by using equation (19) as follows

$$\begin{aligned} \varphi_{y(\mathrm{t})}(\nu) &= \mathrm{E}\left(\exp\left(\mathrm{i}\nu'\mathrm{y(t)}\right)\right) \\ &= \mathrm{E}\left(\exp\left(\mathrm{i}\nu'\left(\textstyle\sum_{\mathrm{m=0}}^{\infty} \mathcal{M}_{m}\ u(\mathrm{t-m}) + \sum_{\mathrm{m=0}}^{\infty} \widetilde{\mathcal{M}}_{m}\ \varepsilon(\mathrm{t-m})\right)\right)\right) \\ &= \textstyle\prod_{\mathrm{m=0}}^{\infty} \mathrm{E}\left(\exp\left(\mathrm{i}\nu'\mathcal{M}_{\mathrm{m}}\, u(\mathrm{t-m})\right)\right) . \prod_{\mathrm{m=0}}^{\infty} \mathrm{E}\left(\exp\left(ir'\widetilde{\mathcal{M}}_{m}\varepsilon(\mathrm{t-m})\right)\right) \end{aligned} \tag{22}$$

where, $i = \sqrt{-1}$. Hence,

$$\varphi_{\mathrm{y(t)}}(\nu) = \prod\nolimits_{\mathrm{m=0}}^{\infty} \varphi_{\mathcal{M}_{\mathrm{m}}\, u(\mathrm{t-m})}(\nu)\ \ \varphi_{\widetilde{\mathcal{M}}_{m}\varepsilon(\mathrm{t-m})}(\nu) \tag{23}$$

where, $\varphi_{\mathcal{M}_{\mathrm{m}}\, u(\mathrm{t-m})}$, $\varphi_{\widetilde{\mathcal{M}}_{m}\varepsilon(\mathrm{t-m})}$ represent the characteristic functions of $\mathcal{M}_{\mathrm{m}}\, u(\mathrm{t-m})$ and $\widetilde{\mathcal{M}}_{m}\varepsilon(\mathrm{t-m})$ respectively.

4. Tsallis Entropy of MCARFIMA System

In this section, we derive the Tsallis entropy for the MCARFIMA model using the characteristic function (28). This approach allows for an analytical evaluation of the system's complexity and uncertainty, providing insights into the underlying stochastic dynamics. Assuming that both the control input $u(\mathrm{t})$ and the noise $\varepsilon(\mathrm{t})$ follow a multivariate Gaussian distribution, with their respective covariance matrices denoted by $\boldsymbol{\Sigma}_{u}$ and $\boldsymbol{\Sigma}_{\varepsilon}$, ensuring that the independence conditions are satisfied, the characteristic function of process y(t) from Equation (19) can be expressed as follows

$$\varphi_{\mathrm{y(t)}}(\nu) = \exp\left(-\frac{\mathrm{i}}{2}\,\nu' \sum_{\mathrm{m}=0}^{\infty}\left(\mathcal{M}_m\,\boldsymbol{\Sigma}_u\mathcal{M}_m' + \widetilde{\mathcal{M}}_m\boldsymbol{\Sigma}_\varepsilon\widetilde{\mathcal{M}}_m'\right)\nu\right) \tag{24}$$

Obviously, the process $\mathrm{y(t)}$ has a Gaussian distribution with covariance matrix

$$\boldsymbol{\Sigma}_{\mathrm{y}} = \sum_{\mathrm{m}=0}^{\infty}\left(\mathcal{M}_m\boldsymbol{\Sigma}_u\mathcal{M}_m' + \widetilde{\mathcal{M}}_m\boldsymbol{\Sigma}_\varepsilon\widetilde{\mathcal{M}}_m'\right) \tag{25}$$

Therefore, the Tsallis entropy of process $\mathrm{y(t)}$ of MCARFIMA system (7) is

$$^{T}H_{\beta}(y) = \begin{cases} \dfrac{\beta^{\frac{d}{2}}}{(1-\beta)}\left(\det\left(2\pi\boldsymbol{\Sigma}_{\mathrm{y}}\right)\right)^{\frac{1-\beta}{2}} - \dfrac{1}{(1-\beta)} & ;\ 0 < \beta \neq 1 \\ \dfrac{1}{2}\ln\left(\det\left(2\pi\mathrm{e}\boldsymbol{\Sigma}_{\mathrm{y}}\right)\right); & \beta = 1 \end{cases} \tag{26}$$

Now, let us consider another case by replacing the Gaussian distribution with the Cauchy distribution. The characteristic function of multivariate Cauchy distribution can be written as follows, Bian (1991) [24]

$$\varphi_{\mathrm{y(t)}}(\nu) = \exp\left\{\mathrm{i}\nu'\mu - \sqrt{\nu'\boldsymbol{\Sigma}_{\mathrm{y}}\nu}\right\} \tag{27}$$

Now, from the equation (28), we have

$$\varphi_{\mathrm{y(t)}}(\nu) = \exp\left(-\sum_{\mathrm{m}=0}^{\infty}\left(\sqrt{\nu'\mathcal{M}_m\boldsymbol{\Sigma}_u\mathcal{M}_m'\nu} + \sqrt{\nu'\widetilde{\mathcal{M}}_m\boldsymbol{\Sigma}_\varepsilon\widetilde{\mathcal{M}}_m'\nu}\right)\right) \tag{28}$$

Given the matrices $\boldsymbol{\Sigma}_u$ and $\boldsymbol{\Sigma}_\varepsilon$ are positive definite, this gives, $\boldsymbol{\Sigma}_u = \mathrm{L}_u\mathrm{L}_u'$ and $\boldsymbol{\Sigma}_\varepsilon = \mathrm{L}_\varepsilon\mathrm{L}_\varepsilon'$ respectively, where $\mathrm{L}_u, \mathrm{L}_\varepsilon \in \mathbb{R}^{\mathrm{d}\times\mathrm{d}}$ .Therefore,

$$\varphi_{\mathrm{y(t)}}(\nu) = \exp\left(-\sum_{\mathrm{m}=0}^{\infty}\left(\sqrt{\nu'\mathcal{K}_m\mathcal{K}_{\mathrm{m}}'\nu} + \sqrt{\nu'\widetilde{\mathcal{K}}_m\widetilde{\mathcal{K}}_m'\nu}\right)\right) \tag{29}$$

where, $\mathcal{K}_{\mathrm{m}} = \mathcal{M}_m\mathrm{L}_u$, $\widetilde{\mathcal{K}}_m = \widetilde{\mathcal{M}}_m\,\mathrm{L}_\varepsilon$. Assume that the matrices $\mathcal{K}_m\mathcal{K}_{\mathrm{m}}', \widetilde{\mathcal{K}}_m\widetilde{\mathcal{K}}_m',\ \mathrm{m} = 0,1,2,\ldots$ are proportional. Then there exists a matrix $\mathcal{D}$ satisfies the following equation [24]

$$\sqrt{\nu'\mathcal{D}\nu} = \sum_{\mathrm{m}=0}^{\infty}\left(\sqrt{\nu'\mathcal{K}_m\mathcal{K}_{\mathrm{m}}'\nu} + \sqrt{\nu'\widetilde{\mathcal{K}}_m\widetilde{\mathcal{K}}_m'\nu}\right) \tag{30}$$

Hence, from Lemma 5, we obtain

$$^{T}H_{\beta}(y) = \begin{cases} \dfrac{1}{(1-\beta)}\left(\dfrac{\beta^{-\frac{\mathrm{d}}{1-\beta}}\,\Gamma\left(\frac{\mathrm{d}+1}{2}\right)\det(4\pi\mathcal{D})^{\frac{1}{2}}}{\sqrt{\pi}}\right)^{(1-\beta)} - \dfrac{1}{(1-\beta)} & ,0 < \beta < \infty, \beta \neq 1 \\ \ln\left(\dfrac{\left(\det(4\mathrm{e}^2\pi\mathcal{D})\right)^{\frac{1}{2}}\Gamma\left(\frac{\mathrm{d}+1}{2}\right)}{\sqrt{\pi}}\right), & \beta = 1 \end{cases} \tag{31}$$

The following algorithm presents the computation of Renyi and Tsallis entropies for MCARFIMA state.

```
Algorithm 1: Renyi and Tsallis entropies for MCARFIMA state.
Require: $d \times d-$ matrices $\boldsymbol{\Sigma}_{u}, \boldsymbol{\Sigma}_{\varepsilon}, \mathrm{C_i}, \mathrm{D_j}, \mathrm{E_k}, \mathrm{i}=1,2,\ldots,\mathrm{p}, \mathrm{j}=1,2,\ldots,\mathrm{r}, \mathrm{k}=1,2,\ldots,\mathrm{q}$ and $0 \le \gamma < 0.5$
Require: $\mathcal{N} >> 1$ and $\beta > 0$.
    Set $\mathcal{M}_0 = \widetilde{\mathcal{M}}_0 = \mathrm{I}_d$
    Calculate $\Omega_m^C$ , $m = 1,2,\ldots.\mathcal{N}$ using equations (11) and (16)
    Compute $\mathcal{M}_m = \widetilde{\mathcal{M}}_m$, $m = 1,2,\ldots.\mathcal{N}$, using equations (25) and (26)
    If $u(\mathrm{t}) \sim \mathrm{MG_d}(0, \boldsymbol{\Sigma}_{u})$ and $\varepsilon(\mathrm{t}) \sim \mathrm{MG_d}(0, \boldsymbol{\Sigma}_{\varepsilon})$, then calculate $\boldsymbol{\Sigma}_{\mathrm{y}}$ and Tsallis entropies using (26) and (36).
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5. Illustrative Example

This section presents an Illustrative example applied to the MCARFIMA system. Consider the following two dimensional input/output system with matrix parameters $\mathbf{C}_1$, $\mathbf{D}_1$ and $\mathbf{E}_1$ taken from [21],

$$\mathbf{C}_1 = \begin{bmatrix} 0.6 & 0.5 \\ -0.8 & 1 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} 3 & -0.8 \\ -1 & 2.2 \end{bmatrix},$$

$$\mathbf{E}_1 = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 1.2 \end{bmatrix}, \quad \gamma = 0.2.$$

Here, $u(t)$ is taken as persistent excitation signal sequence two-dimensional Gaussian distributed with zero mean and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{u}} = \mathbf{I}_{\boldsymbol{d}}$, and $\varepsilon(t)$ as a standard Gaussian with covariance matrix $\boldsymbol{\Sigma}_{\varepsilon} = \mathbf{I}_{\boldsymbol{d}}$. Since $\left(\mathrm{I}_d - \Omega^C(\mathcal{L})\right)$ is invertable, then the process $\mathrm{y(t)}$ satisfies the condition of stability.

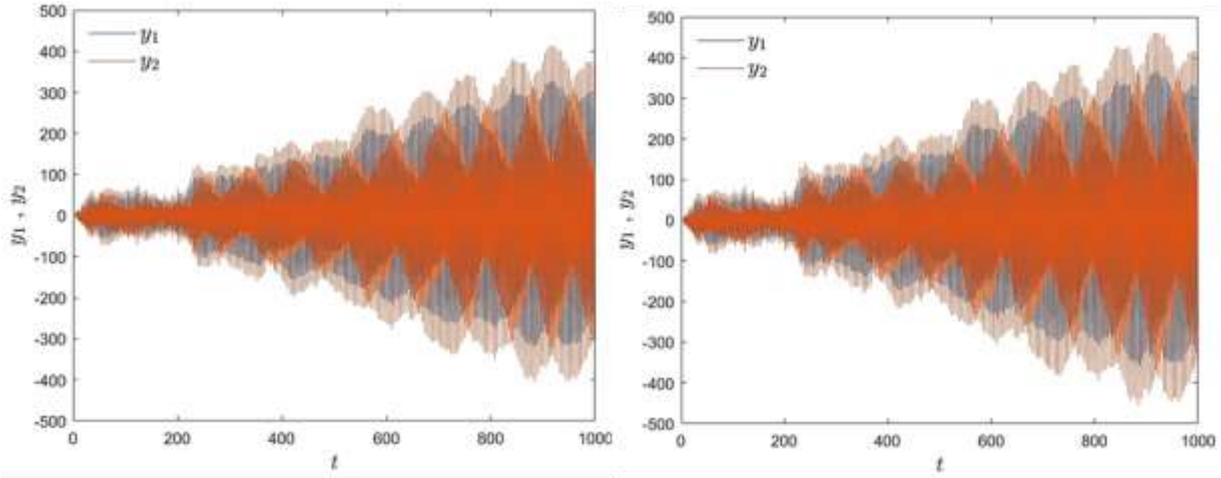

Figure 1: Time evolution of the MCARFIMA system states $y(t)$ for the given parameters C_1, D_1, E_1. The left part of the figure corresponds to $\gamma = 0.2$ while the right part corresponds to$\gamma = 0$

Figure 1 illustrates the effect of the parameter γ on the system states. The left part of the figure corresponds to $\gamma = 0.2$ and the right part corresponds to $\gamma = 0$. As can be observed, varying the value of significantly affects the behavior of the system, influencing both the amplitude and the temporal evolution of the states. This comparison highlights the sensitivity of the system to changes in the parameter.

In Figure 2, we present a comparison for behavior of proposed system using entropy through both types (Renyi and Tsallis). It can be observed that the behavior of Tsallis entropy more flexible compared to the Renyi entropy, which corresponds with the complicated of the system. Additionally, It is worth noting that the Tsallis measure is greater than those based on Shannon and Rényi entropies when $\beta < 1$, but this behavior reverses when $\beta > 1$ becoming smaller . This behavior provides greater flexibility for the Tsallis measure in capturing more details about the random dynamics of MCARFIMA system, making the contribution of this study significant and worthy of further exploration.

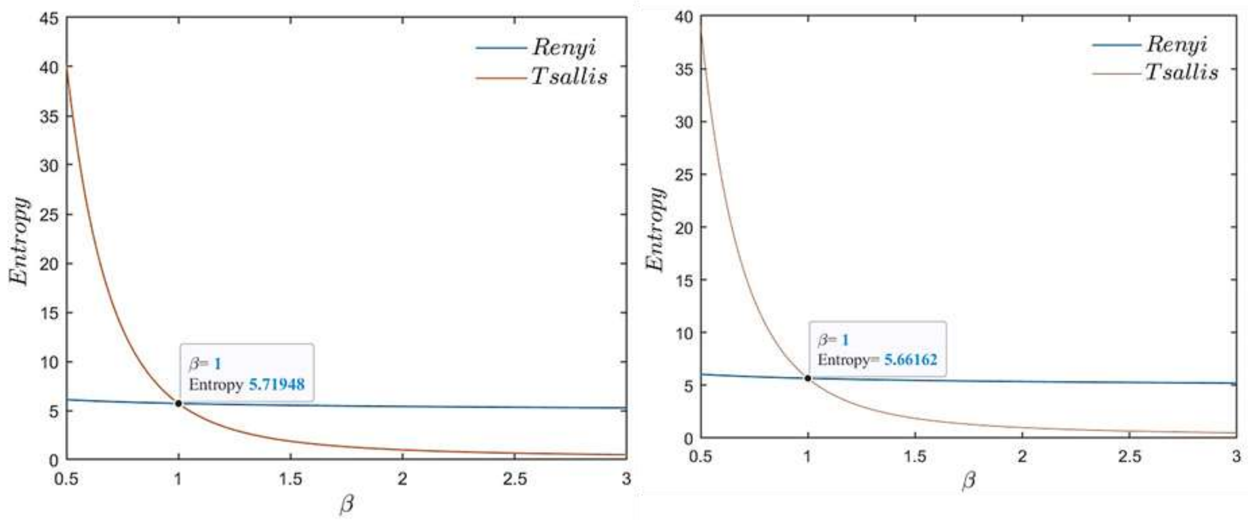

Figure 2 Left: Renyi and Tsallis entropies for MCARFIMA state perturbated by Gaussian white noise with fractional order γ =0.2. Right: Renyi and Tsallis entropies for MCARMA state perturbated by Gaussian white noise

6. Conclusion

In this paper, a general and explicit expression for Tsallis entropy of the MCARFIMA process was derived under two cases (Gaussian and Cauchy processes) using its characteristic function. The covariance matrix of the system state was determined using independence assumptions between the control process and moving average. The results refer that the fractional indicated are highly effective in the system's randomness, highlighting its crucial role in shaping the dynamical behavior. Moreover, conventional methods for quantifying informational uncertainty in time series models primarily emphasize the internal model structure, while often neglecting the influence of output parameters that characterize the system's identity. Therefore, this study contributes a novel perspective by incorporating both input and output parameters into the entropy-based analysis, providing a more comprehensive framework for assessing information content in MCARFIMA systems. Finally, it can be concluded that the Tsallis entropy is more suitable than the Rényi entropy for this type of model, due to its higher flexibility, which allows it to reveal finer details of the information embedded in the system's state.

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منظور انتروبي تساليس في أنظمة سيطرة متعدد المتغيرات ARFIMA

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المستخلص

تتناول هذه الدراسة، انتروبي تسالي لنظام MCARFIMA متعدد المتغيرات، ذي انحدار ذاتي مُتحكم به، ومتكامل جزئيًا، ومُتحرك. وتُشتق دالة مميزة لحالة هذا النظام باستخدام دوالها المميزة المتبقية المقابلة. علاوة على ذلك، يتم اشتقاق صيغة تحليلية لإنتروبيا تسالي لعملية النظام في ظل ظروف مستقلة بين عملية التحكم والمتوسط المتحرك. وتُقترح خوارزمية لحساب قيم إنتروبيا تسالي لعملية النظام عند تأثرها بعمليتي غاوس وكوشي. وتُناقش النتائج وتُحلل لإظهار التكامل الجزئي في البنية المعلوماتية للنظام وسلوكه الديناميكي. وأخيرًا، يُقدم مثال لتوضيح الخوارزمية المقترحة وسلوك المعلومات للنظام المقترح في ظل عملية غاوس.

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